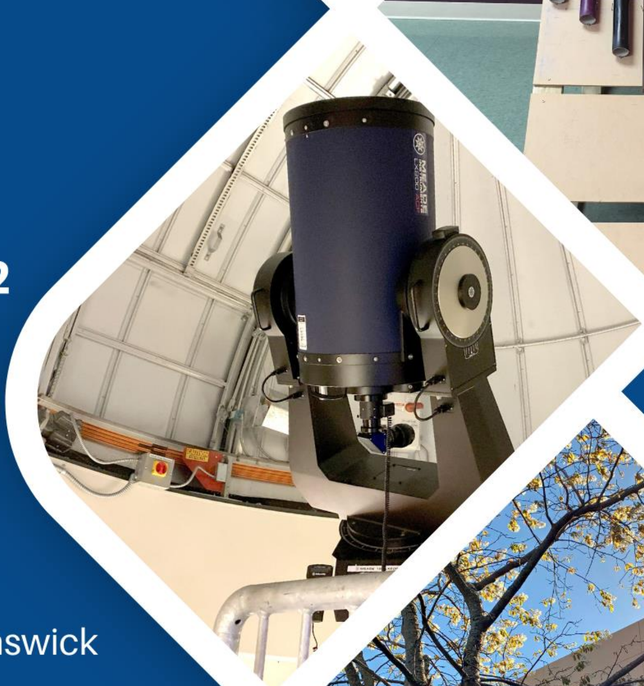
# MACAS 2022

Mathematics and its Connections to the Arts and Sciences

Creating space for mathematics to emerge between, within, and across contexts and disciplines



**Proceedings of the 2022 MACAS Symposium**

**Actes du Symposium MACAS 2022**

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Xavier Robichaud, Olga Fellus, Dragana Martinovic, & Viktor Freiman (eds.)
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Symposium, Shippagan, New Brunswick

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a. Le pavillon Irène Léger, lieu patrimonial du Canada / Pavillon Irène Léger, Canadian Historic Site – Université de Moncton, Campus de Shippagan
b. Marimba en PVC construit par des élèves du primaire d'une école de la région du nord-est du Nouveau-Brunswick / PVC marimba built by elementary students from a school in northeastern New Brunswick
c. Télescope de l'observatoire astronomique / Telescope of the astronomical observatory - Université de Moncton Campus de Shippagan
d. Atrium - Université de Moncton Campus de Shippagan

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# CREATING SPACE FOR MATHEMATICS TO EMERGE BETWEEN, WITHIN AND ACROSS CONTEXTS AND DISCIPLINES

## PROCEEDINGS OF THE 2022 MACAS SYMPOSIUM

HELD VIRTUALLY AT

UNIVERSITÉ DE MONCTON, CAMPUS DE SHIPPAGAN,
SHIPPAGAN, NEW BRUNSWICK

AUGUST 16-19, 2022

EDITED BY

Xavier Robichaud, Olga Fellus, Dragana Martinovic,
& Viktor Freiman

# INTRODUCTION

The symposium series MACAS (Mathematics and its Connections to the Arts and Sciences) was founded in 2005 by an international group of researchers and held for the first time at the University of Education Schwäbsich Gmünd, Germany. Subsequent MACAS meetings were held in Odense, Denmark (2007), Moncton, Canada (2009), Schwäbisch Gmünd, Germany (2015), Copenhagen, Denmark (2017) and Montréal, Canada (2019). In 2022, we broke the tradition of in-person meetings, and organized a fully online symposium, virtually situated at Université de Moncton, Canada.

The vision behind the MACAS initiative is to achieve a humanistic mode of education, that is, to combine various disciplines into a single curriculum, an approach that was suggested by Renaissance philosophers. According to this philosophical view, the goal is to allow students to pursue various fields of study while being introduced to a more holistic perspective, such as connections between mathematics, the arts, and sciences. Given the challenges of the 21st century, interdisciplinary, transdisciplinary, and multidisciplinary education are of increasing importance. In this context, mathematics plays a key role because it is linked to all other disciplines and can serve as a bridge between them.

To achieve their goals, MACAS symposia bring together educators and researchers from diverse fields of study (mathematics, sciences, arts, humanities, philosophy, education, and other disciplines that are intrinsically connected to mathematics), as well as those who are well-established at the forefront of international research and practice, and emerging scholars. The symposia provide a breeding ground for scientific exchange, new partnerships and reflection on commonalities and differences between different contexts, viewpoints, and approaches.

The previous MACAS symposia have shown that there is more than one way to approach these connections in research and in practice. Therefore, this time, we proposed to take a closer look at future challenges and the role of trans-, cross-, and interdisciplinary mathematics education in the Anthropocene era. We invited a wide range of contributions, for example, around the following themes:

- Theoretical study of the relationship between mathematics, arts, and sciences
- Instructional approaches to integrating mathematics, arts, and sciences
- Importance of mathematical modeling and interdisciplinarity for learning mathematics
- Connections between arts and humanities with mathematics in everyday life situations
- Historical and intercultural dimensions of mathematics learning
- Critical analysis of STEM education from a holistic perspective
- Mathematical creativity from an interdisciplinary perspective

The call attracted international participants from around the world including England, Sweden, Denmark, Israel, Russia, Iran, Taiwan, Mexico, Chile, and Canada. Despite their geographic distance, all participants shared an interest in the intersection between mathematics, sciences, and the arts. For these Proceedings, we invited all presenters to submit their full papers. Those submitted were grouped into five broader themes.

**Keynotes**

In her keynote: *Inspired by Mi'kmaw Knowledge: Creating Space for Mathematics to Emerge,* **Lisa Lunney-Borden** shares her decolonizing work that she has been developing alongside Mi'kmaw communities. In this work, she creates space for mathematical ideas to emerge in the context, thus creating greater cultural consistency, rather than cultural collisions. Lisa's work examines stories of mathematics learning that have been influenced by Mi'kmaw knowledge learned from Elders and knowledge keepers in Mi'kma'ki or what we now call Nova Scotia. Through exploration of concrete examples, she shares ideas of how mathematics can live alongside community knowledge systems in ways that support rather than take away from cultural identity, language, and culture.

The next keynote by **Mario Sánchez Aguilar** titled: *Social Media, School Mathematics, and Epistemology*, explores the impact of the Internet and social media on the ways young people interact with and approach school mathematics. This paper highlights how the internet has transformed how students validate mathematical knowledge and seek out mathematical help and support from a range of sources, including online forums and educational websites. The article also explores how social media has provided a platform for students to express their emotions and perceptions about school mathematics in new and innovative ways. The findings challenge traditional notions of being a mathematics student and how mathematical knowledge is acquired and validated.

Two other keynotes were also presented at the Symposium: **Camillia Matuk** keynote titled: *Supporting Youth's Socially Engaged Inquiry Through the Arts* and **Margarida Romero** keynote titled: *Let's Steam! Creative Problem Solving in interdisciplinary Projects*. Please refer to the program for more details: Keynote Speakers.

The section CONTEXTS AND PERSPECTIVES: COMPETENCIES, IDENTITY, AND INTERDISCIPLINARITY features four papers.

**Oliver Kauffmann and Uffe Thomas Jankvist** invite the readers to reflect about the body in discussions of mathematical competencies based on a KOM framework on mathematical competencies. In their turn, **Morten Misfeldt, Uffe Thomas Jankvist, Raimundo Elicer, Andreas Lindenskov Tamborg, Thomas Brahe, Eirini Geraniou and Kajsa Bråting** explore what the interplay between mathematics and computational thinking in K-9 schools in Denmark, Sweden and England looks like through the lens of interdisciplinarity.

By introducing *the Lifestyles Project* consisted in three consecutive assignments (Hobbies, Careers, and Bedroom Design Drawing), **Midhat Noor Kiyani, Limin Jao, Cinzia Di Placido** and **Sun Jung Choi** examine the ways to develop *interdisciplinary mathematics education* initiatives thus transforming mathematics instruction to make it engaging, meaningful, and relevant for the students. In its turn, based on *a framework of figured worlds*, a case from **Rebecca Pearce's** study discusses exploratory findings from interviews with a secondary school student who was born extremely preterm, his parents and teacher to highlight a complexity of *negotiating mathematical identities.*

The section INNOVATIVE APPROACHES TO MATHEMATICS LEARNING: MODELLING, EXPERIMENTS, DESIGN THINKING contains six papers.

It opens with **Amenda Chow's** paper addresses the lack of experiments in the teaching and learning of mathematics, especially at the undergraduate level by suggesting further insights, helpful suggestions, and examples on *incorporating experiments into a university-level* mathematics curriculum. At the other end of K-20 education, a case study on rapturousness in makerspaces presented by **Olga Fellus and Viktor Freiman** features an engineering challenge that kindergarten students from one elementary (K-5) school were trying to solve when designing a shelter for their stuffed animals. Very young children were showing an amazingly complex mathematical thinking while taking a path of development of creativity, perseverance, and more generally, of becoming a well-rounded, inspired, and interest-driven person.

A *virtual lesson study cycle* with high school teachers analyzed by **Gloriana González and Saadeddine Shehab** shows a potential of the *Human-Centered Design* (HCD) approach to engage teachers in identifying authentic contexts for students to experience geometry problem-solving. In their turn, **Olivia Lu, Sreedevi Rajasekharan, and Steven Khan** revisit and resituate *kolam drawing* in mathematics education through a perspective of being for multispecies' flourishing. The authors seek to create opportunities for *passionate immersion and meaningful engagement with other cultures* in ways that privilege a mindset of partnership and kinship.

A paper by **Josh Markle and Jo Towers** describes and interprets students' embodied experiences of spatial reasoning in a grade 12 mathematics classroom using a novel methodology the authors call *Bodymarking to create graphic profiles of everyday classroom actions*, such as gaze and gesture. In his turn, **Bienvenu Rajaonson** demonstrates the potential of the *algorithms created for their application in the learning and improvement of volleyball players' skills*. The examples of *mathematical modeling* discussed in this article present yet another way of raising awareness and advancing research on sports and their integration into society.

The section STEM AND TEACHER EDUCATION includes four papers.

**Amel Kaouche** shares examples of her teaching a modeling course to third-year university students to show the important role of mathematics in solving problems in daily life. In their paper, **Dragana Martinovic and Mariana Milner-Bolotin** propose a novel curricular approach of the Educational Framework for Modelling (EF4M) for integrating mathematics and sciences and highlight the importance of mathematical modelling and interdisciplinarity for teaching and learning STEM. **Heather McPherson** describes how novice teachers assume the role of expert in a dynamic intermingling of roles that can generate pedagogical innovation. **Judith Zamir, Heftsi Zohar and Mark Applebaum** walk us through the first trial for scaling up the Kanga-Kids Training for Math Teachers in early grades. Through the use of a case study, they showcase what works best and what needs to be improved in the scaling up process.

A collection of six texts explores connections between MATHEMATICS, ARTS, AND LANGUAGE.

**Sergei Abramovich and Viktor Freiman** consider mathematics as "creative art" and doing mathematics as "collateral creativity," where visual representation and manipulatives help students to create unexpected and thus exciting solutions. In those cases, the teacher has to react appropriately with encouragement and openness for different approaches.

**Richard Barwell and Yasmine Abtahi** study poetry as a critical thinking tool for mathematics and education. Using the example of modeling and the relational nature of mathematical knowledge and a poem by Ted Hughes and Rumi, the authors seek to demonstrate that the interpretation of particular events through poetry and mathematics reveals mathematical knowledge limitations.

**Viktor Freiman and Alexei Volkov** give us a history lesson on how Leonardo da Vinci and his predecessors calculated the area of a circle. The authors' intent was to introduce us to the historical roots of modern didactical methods.

More beautiful images are provided by **Mohammad Hossein Eslampanah and Payam Seraji**. These were inspired by Persian tiling patterns, thus providing both the cultural and historical perspective on relations between arts and mathematics.

**Revolt Pimenov** takes us into the realm of circle symmetry and uses technology to create beautiful visual representations of geometric problems that extend beyond the boundaries of Euclidian geometry. He provides examples of students' work and inspires us to look for biological forms in mathematical visualizations.

In connection with dance, **Jorge Soto-Andrade, Ami Shulman and May Garcés-Ocares** explore the mathematical processes involved in stochastic dance, a dance created by randomness which generates shapes, forms, movements, and choreography. The authors reveal a reconstruction of a group of dancers' relationship to mathematics triggered by their lived experience of stochastic dance workshop.

There are six texts in the section MATHEMATICS AND SCIENCE AND TECHNOLOGY AND COMPUTATIONAL THINKING.

**Takam Djambong** presents results of a qualitative study with grade 7 and 8 students who participated in tasks addressing Archimedes' Principle, buoyancy, and density using a virtual manipulative environment. Conceptually, the author presents the interdisciplinarity of the tasks through their learning, epistemological, and cognitive components.

Placed in a Danish environment, **Raimundo Elicer and Andreas Lindenskov Tamborg's** study sought to characterize problem handling in the programming and computational thinking-driven mathematics education. Their analysis highlighted three aspects of mathematical problem handling competency in connection to computational thinking, the findings of interest to educators who intend to integrate the two approaches.

**Jacques Kamba and Viktor Freiman** looked into the role of mathematics when elementary school students for the first time used computer coding to program a moving part of a toy. The researchers were particularly interested in understanding the students' engagemenent in debugging during this engineering design project.

**Manon LeBlanc, Nicole Lirette-Pitre and Micaël Richard** address the deficiency of pre-service education as a barrier to providing STEM education in schools. They present results of a two-year long teaching experiment conducted by two teacher educators, who co-taught future teachers by integrating science and mathematics content.

**Dominic Manuel and Marc de Montigny** studied the effects of two novel approaches to teaching physics to undergraduate students, some of whom were future secondary school teachers. Inquiry-based learning is widely considered adequate for teaching science, while flipped classroom was appropriate for blended learning used during the pandemic.

**Yimei Zhang, Tanya Chichekian and Annie Savard** demonstrate how to use four aspects of computational thinking, namely decomposition, abstraction, debugging, and generalization, to empower elementary school students' mathematical problem-solving skills. The authors drew inspiration from problems they found in an ancient Chinese mathematics book "Sun Zi's Mathematical Manual."

## Conclusions and Thanks

With the steadily growing awareness to the necessity and timeliness to surface and showcase the relationship between and among the teaching and learning of mathematics and that of the arts and sciences, we also recognize that education is a science of uncertainty and an art of possibility. In a world that is increasingly typified by volatility, complexity, and ambiguity, the richness of mathematics provides a calming space for exercising hope and developing agency. Mathematics also opens possibilities for understanding multifaceted phenomena through artistic and scientific methodologies. Considering the rich perspectives taken up in the MACAS conference, we also recognize the spaces that are left empty by the duality between mathematics and other school subjects that create a misrepresentation of mathematical content knowledge as siloed, disjointed, and compartmentalized. Such misrepresentations are still prevalent in the public discourse.

We are thankful to all involved in the organization of MACAS 2022, to the Université de Moncton Campus de Shippagan and to our presenters. We want to extend our deepest appreciation for the invaluable assistance of the International Programme Committee: Astrid Beckmann, Viktor Freiman, Uffe Thomas Jankvist, Dragana Martinovic, Claus Michelsen (president) and Annie Savard, the Local Committee: Pierre-Paul Cyr, Lisa Savoie-Ferron, Viktor Freiman, Caitlin Furlong, Patrick Kenny, Manon LeBlanc, Toni Maresu, Hans Peter Nutzinger, Alexandre Pepin, and Xavier Robichaud (president), and Irena Lander for her editing work. Their expertise and dedication in meticulously reviewing and refining the materials have been instrumental in ensuring the quality and accuracy of the symposium and the proceedings.

We are looking forward to meeting again in a location for this time in-person symposium.

Editors February 2024

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# Section 1

# KEYNOTES

# Inspired by Mi'kmaw knowledge: Creating space for mathematics to emerge

Lisa Lunney Borden[1]

*In 2006, Edward Doolittle suggested that rather than imposing mathematics on cultural practices and artifacts, it might make sense to begin in community and pull in mathematics as a need arises. This approach is something that has been a key idea in the decolonizing work I have done alongside Mi'kmaw communities. When we begin with interesting contexts or problems that honour and respect Mi'kmaw knowledge systems, we create space for mathematical ideas to emerge in the context, thus creating greater cultural consistency, rather than cultural collisions. This paper examines stories of mathematics learning that have been influenced by Mi'kmaw knowledge learned from Elders and knowledge keepers in Mi'kma'ki or what we now call Nova Scotia. Through exploration of these examples, I will share ideas of how mathematics can live alongside community knowledge systems in ways that support rather than take away from cultural identity, language, and culture.*

*Keywords: Indigenous mathematics, Mi'kmaw knowledge, decolonizing mathematics, equity.*

## Introduction

This paper highlights some stories that were told as part of a keynote presented at MACAS 2022. In keeping with the inter-disciplinary spirit of MACAS, I have chosen to draw examples from two programs I have been involved with in Mi'kmaw communities: Show Me your Math and Connecting Math to Our Lives and Communities. These two initiatives have grown out of a long-standing relationship I have with Mi'kmaw communities as an educator, researcher, and learner. I begin first by positioning myself in the work to demonstrate how it is that I have come to the understandings I share here. I also do this as a way to honour the Mi'kmaw people who have embraced me and shared knowledge with me. I know that my career today is because of that kindness.

I then share examples from the two programs that highlight the ways in which Mi'kmaw knowledge can be the starting point for learning mathematics and inviting students to engage in mathematical tasks that honour who they are as people. Such an approach creates cultural consistency for Mi'kmaw learners rather than the usual cultural collisions that occur in most classrooms where their knowledge and history is not valued and often ignored completely. It is my hope that these examples might provide the reader with the opportunity to reflect upon their own relationships within Indigenous communities and to consider what possibilities might open up if they begin in community first.

## Positioning myself in the work

As a non-Indigenous scholar who has had the privilege of living and learning alongside Mi'kmaw peoples for over thirty years, I believe it is important for me to acknowledge how I have come to the work and honour those who have helped me come to the understandings I share in this paper. It may seem unusual to begin a paper in this way for many, but in Mi'kmaw traditions it is customary to

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explain where I come from, where I am rooted, and how I am connected within the community. So to that end, I share that I was born and raised (until my teen years) in Saint John, New Brunswick, Canada. I grew up in a part of the city known as the Old North End, a working class and lower socio-economic community where I learned to ride my bike in concrete parking lots and, along with my brothers and the other kids in our neighbourhood, built the best sledding track in the alley behind our home every winter. In the summers, my family would head to a camp on the Kennebecasis river that my grandparents had built in the 1940s following the Second World War. There, I spent days climbing trees, swimming in the river, playing baseball in the field, and running through the woods. Even after we moved from Saint John to a nearby city of Moncton when I was in 7th grade, we still continued to spend our summers at the camp in Chapel Grove. It is the place where I feel most rooted. I can close my eyes and walk the grounds in my mind's eye. I know every tree, every rock, the sound of the birds in the morning, the smell and feel of the river and the lands that surround it. It is home.

Decades later, as a teacher in We'koqma'q First Nation, I would have the opportunity to read part of a Mi'kmaw creation story describing how Kluskap, being annoyed with Beaver, went to find him in the Kenebecasis River where he lived in his home on Long Island, and chased him up the Wulastuk River, now known as the St. John River. Long Island is located directly across the water from our family's beach. Every year, we would take a boat trip with a family friend to spend the day on the largely uninhabited island, swimming on the sandy side of the river and climbing through the trees and up the rock faces. How is it that I was learning something completely new about a place I knew so well? This moment reminded me that as a settler on these lands, even one whose family has been here since settlers first came to live on these lands, there are so many stories I do not know. There were tens of thousands of years of stories happening long before our arrival. As we consider the ubiquity of often empty land acknowledgements that are commonplace at conferences and in institutions, I am always aware of my treaty obligations to continue to learn about this place I now call home and to be ever mindful of the relatively brief history my ancestors have had in this place.

## A teacher and a learner

As a university student in both my undergraduate degree and my education degree, I volunteered with X-Project, a student society at St. Francis Xavier University (STFX) that supports Mi'kmaw and African Nova Scotian youth with education, recreation, and leadership programs. During that time, I had many opportunities to build meaningful relationships with community members and learn about how the education system failed to serve Mi'kmaw and Black youth. I still recall a community Elder telling me that she wished that the youth in her community had their own school. She stated emphatically, "I'm not prepared to sacrifice another generation of children waiting for the school system to change." When I became a teacher in a Mi'kmaw community school, I carried those words with me. What would it mean to teach in a way that honoured her wish? How could I disrupt the status quo to ensure Mi'kmaw children were being well-served by their community school? This became a guiding philosophy for my teaching career.

In 1995, I began working in We'koqma'q First Nation, a Mi'kmaw community along the shores of the Bras D'Ors Lakes in Unama'ki (Cape Breton). I taught grades 7 to 12 mathematics, and other subjects as assigned, for 10 years. I was consciously a learner in that role, knowing that while I knew some things about mathematics teaching, I also had a lot to learn about the community, the people, the history, the language, the culture. I haven't the time or space in this paper to talk about all the ways in which the community supported and embraced me and helped me to do that learning —and continues to do so this day—but I am forever grateful for the opportunities I have had and know that whatever success I have in this career is in large part due to that generosity (see Lunney Borden, 2016). Learning to speak Mi'kmaq was an integral part of my learning journey that has shaped how I teach mathematics (see Lunney Borden, 2011, 2013) and learning from Elders and knowledge keepers has helped me to see that mathematical thinking is so much more than what is depicted in textbooks.

During my time in We'kqoma'q, it would become part of a collective of Mi'kmaw communities working together for the education of Mi'kmaw youth. This collective, known as Mi'kmaw Kina'matnewey (MK), would support the capacity building of communities to ensure local people were trained as teachers and administrators so that there could be true community control of education (Paul et al., 2019). MK advocates for Mi'kmaw education rooted in language and culture and serves as the collective voice for all partner communities. Since the MK agreement was signed with the federal government in 1998, graduation rates have increased to approximately 90% annually and post-secondary enrollments are ever increasing. My ongoing relationship with MK, now as an academic, allows me to regularly work with teachers and their students in classrooms constantly seeking ways to improve the educational experiences for Mi'kmaw youth in mathematics.

## Indigenous knowledge matters

I am not alone in my desire to transform mathematics for Indigenous youth; decades of government reports and mandates have called upon educators to better serve Indigenous youth in all areas of education. The first wave of calls to ensure Indigenous control of Indigenous education came in response to the 1969 Trudeau government *White Paper on Indian Policy* (Government of Canada, 1969), with the subsequent Indigenous resistance to assimilation that was described in *Indian Control of Indian Education* (National Indian Brotherhood, 1972). Following this call, scholars began to advocate for education systems to attend to Indigenous knowledge systems and value the ways of knowing inherent in languages (Battiste, 1987). Later, scholars would bring this lens to STEM teaching and learning in Indigenous contexts (Cajete, 1994; McIvor, 1995; Mount Pleasant-Jetté, 1998), and point to the importance of attending to tensions between Indigenous and non-Indigenous ways of knowing, being, and doing (Aikenhead, 1996; Lipka, 1994). This eventually led to provincial/territorial mandates to integrate Indigenous perspectives in K-12 curricula (Aikenhead & Elliott, 2010) that began to appear around 2000 (Wiseman, 2016). This often resulted in mathematics being applied to Indigenous contexts in textbooks.

Students are asked to create a linear system to write equations to describe the perimeter of a Métis flag, to write an equation to determine how many of the 545 cones Talise has on her jingle dress if she has 185 more than her sister, and to create a linear system to determine how many 6-stone or 7-stone Inuksuits were sold (presumably to tourists) by a store (Pearson Canada, 2010). All of these examples came from one unit in the textbook currently being used for Grade 10 in Nova Scotia. Measuring a flag does not recognize the mathematical thinking of the community represented by that flag, it is simply a measurement task. Counting cones on a jingle dress does not help students to learn about the dress or the role it plays in ceremony. A jingle dress is a prayer dress that is worn by a dancer to dance for healing, and many teachings of the jingle dress is that it always has 365 cones, one for each day of the year. Talise's sister has an unfinished dress. Inuksuits are integral to Inuit to navigate the Arctic, but I cannot help but wonder if Inuit were consulted of the question advocating selling them to tourists. These types of questions demonstrate how textbook companies tend to take a surface level approach to the mandates to teach Indigenous perspectives in all subjects. None of these questions recognize the wealth of mathematical thinking in Indigenous knowledge systems, they simply apply typical school-based mathematics to images from Indigenous communities, and seemingly do so without any consideration of the sacredness of those artifacts.

In 2006, in a plenary address to the Canadian Mathematics Education Study Group, Edward Doolittle, a Mohawk mathematician, argued that rather than imposing mathematics on cultural practices and artifacts, it might make sense to begin in community and pull in mathematics as a need arises. This begins by taking seriously Indigenous ways of knowing, being, and doing, and the knowledge systems that are inherent in these communities. Such approaches have led to promising practices that involve beginning in place (Zinga & Styres, 2011), and allowing mathematics to emerge from interesting contexts rooted in community knowledge systems (Lunney Borden & Wiseman, 2016). Following the Truth and Reconciliation Commission's 2015 release of the report, even more attention is being paid to addressing the need for Indigenous knowledge to have a rightful place in education across all subjects and grades.

As a mathematics educator, I recognize there is a need to question mathematics itself, what counts as mathematics, and who gets to decide. Joseph (2010) has argued that ideological beliefs about European superiority meant that "The contributions of the colonized peoples were ignored or devalued as part of the rationale for subjugation and dominance" (p. 4) in mathematics as they were in other subjects. This provokes me to question whose mathematics are we teaching and why? I align my thinking with Gutiérrez (2017) who claimed that "School mathematics curricula emphasizing terms like Pythagorean theorem and pi perpetuate a perception that mathematics was largely developed by the Greeks and other Europeans" (p. 17). Should we continue to perpetuate the myth that Pythagoras discovered the theorem often attributed to him when there is considerable evidence to show that this theorem was known by Babylonians thousands of years before his birth? Were the Greeks the only people to be interested in the relationship between the circumference and diameter of a circle? I will share that similar knowledge was passed down through generations of Mi'kmaw people in a later section of this paper. The choices we make in mathematics teaching and learning can either disrupt these myths of white superiority or reinforce them. For me and my work, I have regularly chosen the disruptive path and I believe it has made all the difference.

In this paper, I share two key ideas that have emerged for me in reflecting upon the work I have done over my career as an academic. While these do not represent an exhaustive list of how to better address the learning needs of Indigenous students in mathematics, they are two ideas that I believe best reflect the goals of MACAS in that they align with ideas of connections across mathematics, science and the arts. First, I will discuss the ways in which I have come to understand the importance of Elder knowledge and then I will describe the role of ethics in considering mathematics teaching and learning experiences. In both instances I will share some examples from work I have done in schools and with pre-service and in-service teachers in various programs.

## The significance of elder knowledge

As a teacher, I would often say to my students that there was mathematical thinking within the community, it just didn't get written down in the textbooks we use in school. Many members of the community were known for their ability to make baskets woven from wood, typically black or white ash, often dyed to add colour and occasionally embellished with strands of sweetgrass. Their creations were beautifully elaborate, with finely woven strips and curled decorative points (see Figure 1) that clearly involved knowledge that would align with what we might describe as mathematical thinking.

**Figure 1: Mi'kmaw woven tea cup and basket**

Other community members were known to make beautiful beadwork for regalia, earrings, medallions, and other beautiful works of art. Mi'kmaw people were also known for making snowshoes, canoes, axe handles, and hockey sticks, all work that involved processes we might describe as mathematical. Our textbooks never recognized this sort of mathematical knowledge or, if it did, what was presented was often trivializing merely applying Western mathematics to these artifacts rather than considering the Mi'kmaw knowledge systems that were apparent in the creation of these objects. Consider for example, that Mi'kmaw people had canoes that could travel from Cape Breton to Newfoundland. Many Nova Scotians and Newfoundlanders will tell you that a decently gusty wind will prevent the ferry from making its way across the Cabot Strait on any given day, so one must wonder about the technology involved in creating a canoe that can accomplish this task. Yet our education system repeatedly fails to acknowledge the ingenuity and innovation in Indigenous knowledge systems, nor does it recognize these knowledges are powerful opportunities for learning what we might today refer to as STEM education. As an educator I wanted my students to know that STEM, and mathematics in particular, has always been a part of their heritage.

Early in my academic career, as a young doctoral student, I was excited to have the time and space to talk with Elders I had known for years, to have conversations about how we could make more explicit the mathematical thinking that was such a part of Mi'kmaw ways of knowing, being, and doing. I was very fortunate that my doctoral advisor, Dave Wagner, had recently received funds to support these sorts of conversations. He was interested in ethnomathematics and how mathematics was being used in out of school contexts. I was interested in talking with Mi'kmaw Elders. This was a perfect match. We began our conversations online with a group of Elders and language teachers who have gathered at a community school to join us online as poor weather had prevented us from going in person (See Wagner & Lunney Borden, 2015, for a more detailed discussion). One Elder who joined us that day was the now late Dianne Toney. Dianne was a quill box maker. She made boxes from birch bark, wood strips, and porcupine quills (Figure 2). She explained how she always began her box with a circular top made from birch bark. She would use a wood strip, similar to those used for basket making, to make the side of her box. She would then use an awl to poke holes in the bark and weave her porcupine quills through the bark to create her patterns. She told us that to make the ring go around the circular top, she would measure three times across the top and add a thumb width and it would make a perfect ring every time. I recall anxiously exclaiming that this was "Pi!" to which she replied that it was common sense. She explained how she had learned this from generations of quill box makers who had taught her (Wagner & Lunney Borden, 2015). She stated that it was important to have a strip that was long enough but not too long to have waste; three and a thumb width gave her just this right amount.

**Figure 2: Porcupine quill box made by Dianne Toney**

Being able to determine the right amount without waste is an important part of Mi'kmaw ways of knowing, being, and doing. This value is reflected in the concept of netukulimk which loosely translates to sustainability but captures a deeper sense of responsibility to all one's relations to ensure that one takes only what is needed for survival and, in so doing, ensures the survival of others as well including non-human relatives. The mathematics Dianne used emerged from this value of netukulimk and in response to the need to figure out how much was enough. Much of what we teach in school mathematics has also come from a need to answer questions, but those questions have often been rooted in very different value systems.

## Show me your math

The conversation with Dianne was so inspiring that Dave and I knew we could not be the only people engaging in these conversations. We wanted youth to be able to have these sorts of conversations with Elders and knowledge keepers. After a few conversations together and then with teachers, administrators, and Elders in Mi'kmaw schools, the Show Me Your Math (SMYM) program was born (Lunney Borden et al., 2019). SMYM invited Mi'kmaw youth to have conversations with Elders in their own community context to learn about the mathematical thinking that has always been a part of Mi'kmaw ways of knowing, being, and doing. The program ran from 2007 until 2017 and hundreds of projects, including numerous classroom-based inquiry projects, emerged as a part of SMYM. Each year one of the MK community schools hosted the annual SMYM Math Fair where hundreds of children would share their learning with their peers. The projects themselves took the learning far beyond mathematics and allowed children to see how a question about mathematics can lead to learning about culture, community, and the impacts of colonialism. Dianne did not live long enough to see the legacy that her idea inspired. In May of 2006, as I was preparing to go have a follow up conversation with her, I received a phone call from a former colleague and friend telling me that she had passed away through the night of a heart attack. I like to think of SMYM as her living legacy of teaching children about the vast knowledge within their own cultural communities.

## Waltes

Waltes is a Mi'kmaw game of chance in which players score points by banging a bowl filled with two-sided dice made from bone. When 5 or 6 of the 6 dice turn up on the same side, the player scores points and can collect sticks. If the player scores multiple points in a row, they can earn the notched sticks known as the old lady or old man. The aim of the game is to collect all the sticks by continuing to play through a series of rounds during which the counting rules change. The counting system is quite complex and often Elders are used to support the counting as younger people play the game. Waltes became a popular topic to explore in SMYM. Some participants focused on the probability of scoring a point or multiple points in a row, others focused more on the general rules of the game and the role of counting in the game.

In Gallagher-MacKay and Steinhauer (2017), former participant, Aaron Prosper, described his experiences with SMYM and talked about learning waltes from his grandparents. In addition to learning about the game, he also became curious about why there was a hole drilled into the bottom of the waltes bowl. This was when he learned about how Indian Agents, representing the Government of Canada, would come to Mi'kmaw communities aiming to enforce colonial policies outlined in the Indian Act (1876). The Indian Act "sought to place First Nations individuals and communities, their lands, and their finances under federal government control" (TRC, 2015, p.110). A part of the Indian act that was in place between 1884-1951, commonly called the potlatch ban, prohibited cultural ceremonies (Joseph, 2018), which prevented Indigenous people from passing on culture and traditions. Waltes became a target for Indian agents who feared the Mi'kmaw would use the bowls for ceremony and drilled holes in them to prevent them from holding water. In many families today, there are waltes bowls that still carry that scar of colonialism. Aaron talked about how he was learning so much more than the mathematics involved in waltes, participating in SMYM allowed him to explore this game in a much more holistic way making connections to his family history and the history of his community.

## Birch bark biting

Birch bark biting became an important part of SMYM as well. Although this was not an early project it became a widely celebrated project later in the years of SMYM. The idea for working on birchbark biting actually came from a conversation with an elder. We were discussing ideas that we could use in mathematics for young children. In a research discussion, she told me, "When I was a young girl, my mother used to peel thin strips of bark off the logs and ask us to fold them and bite shapes into them." Naturally, my curiosity was peaked. I asked her more about this as I knew of birch bark biting that happened elsewhere in other Indigenous nations in what we now call Canada, but I had not known it was something commonly done in Mi'kma'ki. She explained that it had in fact been a common past time in her childhood, but she was not sure if anyone still was able to do it.

I took out some paper, as we had no bark available at the time, and asked her to show me how to fold the paper to do the birch bark biting. She instructed me to fold it in half, and then rotate it and fold it in half along that first fold. As I worked to line up the paper, I asked her if there was a Mi'kmaw word to describe this process. She replied, "Yes! Tetpaikatu!" I asked, "What does that mean?" to which she told me "Fold it the right way!" and we both laughed. She suggested that I learn more about it by doing some research and maybe students would want to learn it too. I did just that.

In my searching for information about birch bark biting, I came across an article written by Oberholtzer and Smith (1995), two anthropologists, who had travelled the country interviewing people who were known to be birch bark biters, each of whom believed they were one of the last people in their communities who could still do these bitings. As I read the article and came to a paragraph on the second page, I was stopped in my tracks. There was a passage about Margaret

Johnson of Eskasoni, a basket maker and birch bark biter. I knew Margaret Johnson, or Dr. Granny, as she was commonly called throughout Mi’kma’ki. I knew her family, had taught some of her grandchildren at the university, and knew her sister, Caroline Gould, very well as she was an Elder and basket maker in We’kqoma’q First Nation where I had lived and worked. As I reached the bottom of the paragraph, I saw the line that stated her sister, in another community, was also a birch bark biter. I knew that must have been referring to Caroline. Unfortunately, by the time I found this article both women had passed on to the spirit world, but knowing their history with birch bark biting, we knew it was something students should learn about.

I set up a plan with a teacher in one of our Mi’kmaw schools and we decided to go work with grades 5 through 8 to teach them about birch bark biting and to try it for ourselves. We had collected some bark but knew we would need more. One of the teachers at the school contacted someone in the community who could bring us more bark. When he arrived with a great big barrel of birch bark, I asked about it. He told me he had collected it up the mountain when people were logging up there. He would go and harvest the bark. He told me that he had collected this bark years ago for Dianne, but she never got to use it. I took this as a sign we were on the right path.

We figured out how to do birch bark biting by watching videos online of other birch bark biters and working together to figure it out. The students took to it instantly and found it enjoyable and engaging. We were impressed with the work that they were able to create (see figure 3).

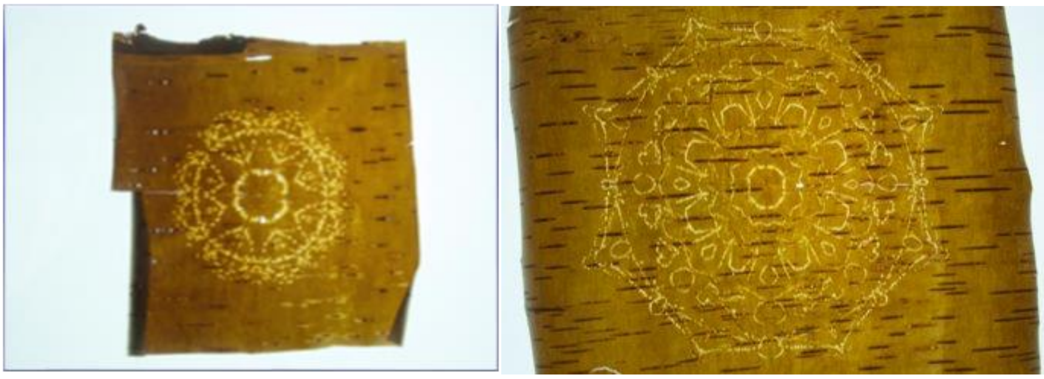

**Figure 3: Student birch bark bitings**

While it might be tempting to impose mathematics on the artifacts as they are depicted, where the real mathematics happens is in the creation of these images. One must really understand that a circle is a collection of points that are all equidistant from a centre to create a circle when only biting one small part of it and then unfolding the bark to create a full design. The 8-point star was also something that required significant understanding of angles to create. The student who created the 8-point star in figure 3 had worked for some time on getting it to be just right. He had to think about the angles he was employing and how the paper was folded. There is a significant focus on the role of visualization in creating birch bark bitings. We also noted that through folding the paper, students became very aware of the fractions involved and were able to easily explain halves, quarters, eighths, and sixteenths that came from folding the bark.

In addition to the mathematical thinking that was emerging from the work done with birch bark biting, students were also learning stories about the people in their communities who had done it as a practice. Stories were being told by teachers who were remembering seeing it happen as a child or hearing stories about it from Elders. Students were also learning more about birch trees and the uses of birch bark and the proper way to collect bark. Again, the learning extended far beyond the math.

## Ethics and social justice: Connecting math to our lives and communities

After years of show me your math, we had many communities wondering about offering it to their students who were not attending MK schools. Working with colleagues, we began an outreach program called connecting math to our lives and communities. In this program we decided to focus on social justice mathematics activities or activities that allowed students to see the power of mathematics to read and write the world (Gutstein, 2006). We wanted youth to see that they could tell their own stories using mathematics as a tool. This program was expanded to also include local African Nova Scotian communities. We drew inspiration from issues impacting the communities we were serving such as climate change, soil erosion, environmental racism (Waldron, 2021), and water insecurity to name but a few. Water security became one of our first modules following a conversation with a teacher in one Mi'kmaw community who shared that they were once again on a boil water order in the community and that she thought it might make for a good SMYM project to raise awareness about this issue.

Drawing from data obtained from the Halifax Regional Municipality's water authority, we developed a series of activities that would allow students to determine how much water their household uses in a year. The data was provided in cubic metres and gave the average water use for a 3-montn period based on the number of people living in a home. This created numerous opportunities to solve problems that would require some multiplicative thinking and proportional reasoning. We had students physically construct a cubic metre and then determine how many litres that represented (Figure 4). This provided students with a good visual to imagine how much water would be needed for a whole community. It also allowed us to model how contaminants, often measured in parts per million (ppm), can be relatively small and still impact an entire water system. Using base-ten materials or Dienes blocks, we recognized that the cubic metre is the size of the million cube if the small cube represents 1 unit. Therefore, the unit cube would be one-millionth of the cubic metre or one ppm.

**Figure 4: Students building a cubic metre**

This was one of many examples that show how mathematics can help youth to understand and talk about important issues that affect their community. We have engaged in numerous other projects that look at the impacts of hurricane force winds, the disruption to an ecosystem when an invasive species is introduced, and the role of mathematics in understanding matters of wealth inequality or land back. All of these moments allow students to see that mathematics is a useful tool for understanding important issues in our society. They allow students to consider the ways in which we use mathematics and to discuss if we are being ethical in that use.

## Concluding thoughts

For far too long, many children have endured mathematics learning experiences that were not designed for them and were not reflective of their lived experiences. The examples I have described in this paper are part of my attempts to address this disconnect. Papachese Cree scholar Dwayne Donald has argued that "colonial logics" (Donald, 2009, p. 7) divide people and peoples from their relations instead of opening a space where they might co-exist (Donald, 2012). Colonial logics have been pervasive in mathematics and the projects of SMYM and CMTOLC have been designed in a way that aims to open up the kinds of space Donald hopes for. The work I have done alongside communities comes from long-standing relationships of listening and learning together to bring the community's knowledge into the classroom's learning experiences.

The stories I have shared are but a few examples of how a commitment to respect, reciprocity, relationship, and relevance (Kirkness & Barnhardt, 1991) allow for interesting mathematics to emerge in ways that allow learners to learn mathematics while also learning about their own cultural identity. Such an approach allows students to learn in a culturally consistent way. I end with the following quote from an article I wrote with my research partner Dawn Wiseman, inviting reflection from the readers:

> Our intent in teaching and learning is not to begin with STEM expectations or outcomes but rather to begin in a place we knew had the potential to teach. Though we cannot guarantee that STEM will emerge, we know the potential is in place. In each of our stories, the activities opened up spaces from which explorations, questions, and conversations could emerge and live for a while. When these spaces open up, what we find is important is taking the time to be with what they teach, to pay keen attention to the possibilities for teaching and learning. In this way, we see STEM as an artifact of teaching and learning, not a framework imposed upon it. (Lunney Borden & Wiseman, 2016, p. 150)

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# Social media, school mathematics, and epistemology

Mario Sánchez Aguilar[1]

*This article explores the impact of the Internet and social media on the ways young people interact with and approach school mathematics. Drawing on empirical research, the article highlights how the internet has transformed how students validate mathematical knowledge and seek out mathematical help and support from a range of sources, including online forums and educational websites. The article also explores how social media has provided a platform for students to express their emotions and perceptions about school mathematics in new and innovative ways. Through analysis of online activity, the article reveals that students use the internet to seek clarification on mathematical doubts and solve mathematical tasks. The findings challenge traditional notions of being a mathematics student and how mathematical knowledge is acquired and validated.*

*Keywords: Social media, epistemology, school mathematics, emotions, mathematical help-seeking.*

## Introduction

For some years, I have been researching how young people use the internet and social media in connection to school mathematics. This research has focused on using the internet as a source of mathematical help and a platform for sharing opinions and feelings about the subject.

This research suggests that the internet and social media have significantly impacted how students interact with and approach school mathematics. Through their online activity, students can now share their perceptions about the nature of school mathematics in new and innovative ways. They are also able to seek out mathematical help and support from a range of sources, including online forums and educational websites.

Indeed, the internet has transformed how students validate mathematical knowledge. Rather than relying solely on traditional sources of authority, such as textbooks or teachers, students increasingly turn to online communities to confirm and verify the accuracy of mathematical information (e.g., van de Sande, 2011). This trend is a significant development in mathematics education. It challenges traditional notions of being a mathematics student and how mathematical knowledge is acquired and validated.

In this article, I report on some of these findings, focusing on two issues:

- How people use a social network to express their emotions and perceptions about school mathematics.
- How students use the internet as a source of mathematical help to clarify their doubts and solve mathematical tasks.

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## Using a social network to express emotions and perceptions about school mathematics

School mathematics is a subject that is commonly accompanied by an array of emotions, be they positive or negative, as observed by scholars such as Pepin and Roesken-Winter (2005). These emotions are of particular significance, as they form the cornerstone upon which individuals establish their relationship with and self-identify in school mathematics. Furthermore, attitudes and beliefs towards the subject play a pivotal role in shaping such emotions and are thus integral to understanding the complex nature of human interactions with school mathematics.

On the other hand, social media platforms allow individuals to share and express their emotions towards various aspects of their lives, such as work, personal relationships, food, exercise, pets, and school (Stieglitz & Dang-Xuan, 2013). Mathematics is not exempt from this phenomenon as a subject that often elicits strong emotions of love or hate. Twitter is one of those social media platforms where individuals express their emotions related to school mathematics. However, the potential role of this platform as a window into the emotions that people associate with this subject is not commonly recognized in the specialized literature.

Twitter has been proposed as a means to engage mathematics students in and out of the classroom (Soto & Hargis, 2017), and studies have shown that it positively impacts students' learning of mathematics (Vohra, 2016). Furthermore, Twitter has been used as an organizational tool to manage classroom issues, such as reminding students about assignments and upcoming tests (Danesi, 2016). Additionally, Twitter has been suggested as a space for exchanging ideas, dialogue, discussion, and interaction within the community of mathematics education research (Chernoff, 2014). Nevertheless, this article aims to support the assertion that Twitter can serve as a "mood indicator" in relation to school mathematics (Danesi, 2016). Specifically, Twitter can be employed to monitor and examine the moods and attitudes of individuals toward various subjects, including school mathematics. By scrutinizing tweets related to school mathematics, researchers can acquire valuable insights into the general public's perception of this subject.

In order to advance this argument, a brief analysis of how mathematics is represented in this social network is introduced; in particular, a categorization of users' tweets about school mathematics is presented. Such categorization provides insight into the emotions and perceptions that people associate with school mathematics nowadays.

## Choosing and categorizing tweets

Twitter is awash with tweets relating to mathematics, such as popular articles, images, and videos about mathematical curiosities, announcements of conferences and academic events by organizations and their members, and publicity by scientific companies for new articles and journal issues. Simply searching for "mathematics" in Twitter's search engine confirms this. However, the following categorization focuses on tweets where authors express their positive or negative opinions or some form of sentiment towards mathematics or its related subjects. This categorization includes emotions such as sympathy, dislike, and confusion.

Over the past years, I have collected a group of tweets by locating them in various ways (see Aguilar, 2021). These tweets were either in my timeline, retweeted by colleagues and friends, or found through monthly keyword searches using terms like "math," "mathematics," and "matemáticas." As a result, I created a collection of 88 tweets, mainly in English but some in Spanish. Most of these tweets focus on school mathematics because the authors are likely to encounter mathematics in that setting. Additionally, many tweets are written in a humorous or satirical style. Notably, some of these tweets have received significant support, as evidenced by hundreds or thousands of "likes." This appraisal could indicate that many users identify with or appreciate the content of these tweets.

To categorize the 88 tweets, the technique of constant comparison, as described by Teppo (2015), was implemented. This process involved creating codes for each tweet, such as "tweets about struggling with mathematics" or "tweets regarding difficulties with mathematics homework." These codes were subsequently merged into five overarching categories. The resulting categories are as follows:

- Mathematics is difficult
- Mathematics is useless
- Mathematics tests
- I like mathematics
- Love and mathematics

In the next section, a brief description of each category is provided, along with an accompanying image corresponding to a tweet.

## Emotions about mathematics

The tweets showcased in this section include the publication date, the username of the author, the number of likes and retweets they have received, and a link for accessing them. An English translation is provided if a tweet was originally posted in Spanish.

## Mathematics is difficult

This category comprises tweets where users express the challenges of comprehending mathematics and the emotions, such as frustration, associated with such difficulties. Additionally, it includes tweets that depict mathematics as a complex and mentally demanding topic (Figure 1).

*Publication date*: July 10, 2016
*Statistics*: 11 retweets; 20 likes
*Translation*: When everyone understands the definition of continuity except you


**Figure 1: Tweet by @Infinito307 retrieved from https://twitter.com/Infinito307/status/752341078523596804**

## Mathematics is useless

This category includes tweets claiming mathematics is useless in everyday life or work (Figure 2). For instance, some tweets devalue mathematical knowledge compared to skills such as writing a CV or understanding how to pay taxes. One example is a tweet that states, “Another damned day without using algebra.”

*Publication date*: March 5, 2014
*Statistics*: 2,172 retweets; 1,540 likes

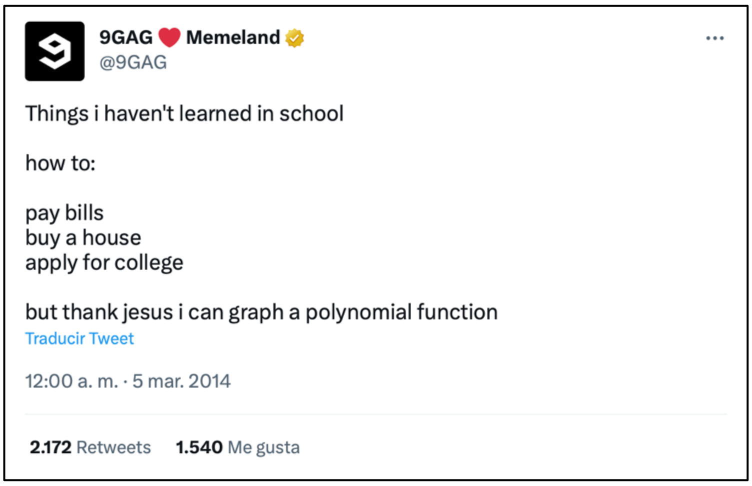


**Figure 2: Tweet by @9GAG retrieved from https://twitter.com/9GAG/status/441090800676777984**

## Mathematics tests

Assessment is a critical element in the academic success of mathematics students. Numerous tweets express students’ perceptions and experiences related to mathematics tests, such as difficulty level, unfair assessments, or unrealistic contexts in which problems are posed (Figure 3). For example, mathematical problems may refer to semi-reality, which some students find problematic (Skovsmose, 2001).

*Publication date*: February 13, 2014
*Statistics*: 502 retweets; 349 likes

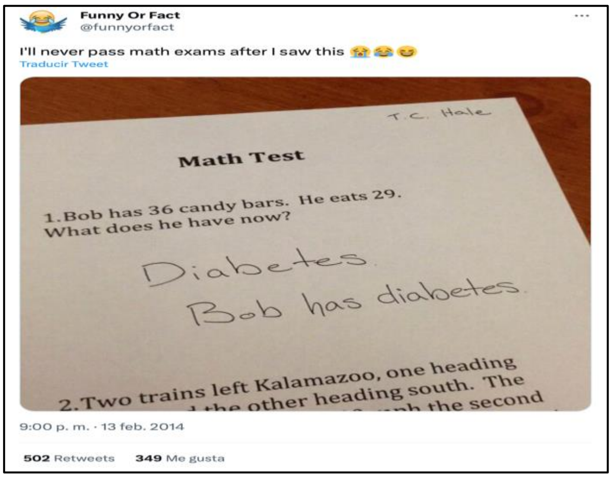


**Figure 3: Tweet by @funnyorfact retrieved from https://twitter.com/funnyorfact/status/434160175592009729**

## I like mathematics

Although individuals share tweets expressing positive emotions towards mathematics, these tweets are not retweeted or favorited as frequently as those belonging to the abovementioned categories. While some people explicitly state the positive emotions that mathematics evokes in them, others express their affection for mathematics without providing any additional context (Figure 4). Some employ more imaginative means to convey their admiration for mathematics.

| | |
|---|---|
| *Publication date*: | October 10, 2018 |
| *Statistics*: | 1 retweet; 1 like |

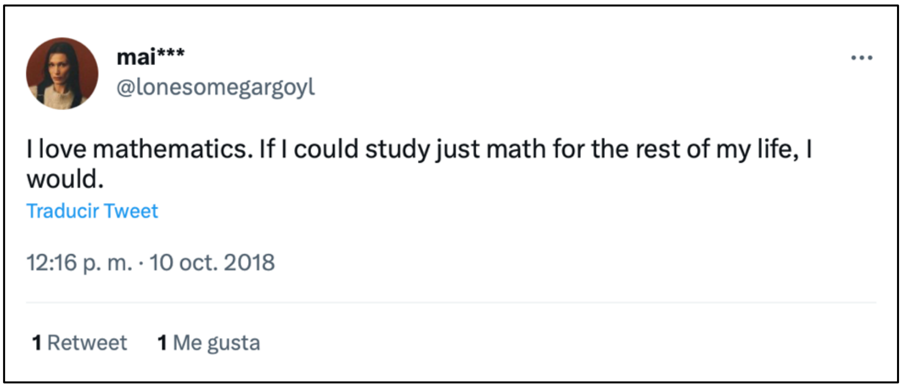


**Figure 4: Tweet by @lonesomegargoyl retrieved from https://twitter.com/lonesomegargoyl/status/1050072549185466369**

## Love and mathematics

Indeed, some individuals tweet about both love and mathematics; however, their tweets do not pertain to a love for mathematics. Instead, these people tweet about romantic love, which may include sentiments of heartbreak, and attempt to draw connections to mathematics. An instance of such a tweet is: “I understand multivariable calculus, but I do not understand life without you” (Figure 5).

| | |
|---|---|
| *Publication date*: | December 24, 2014 |
| *Statistics*: | 8 retweets; 45 likes |
| *Translation*: | I understand multivariable calculus, but I do not understand life without you |

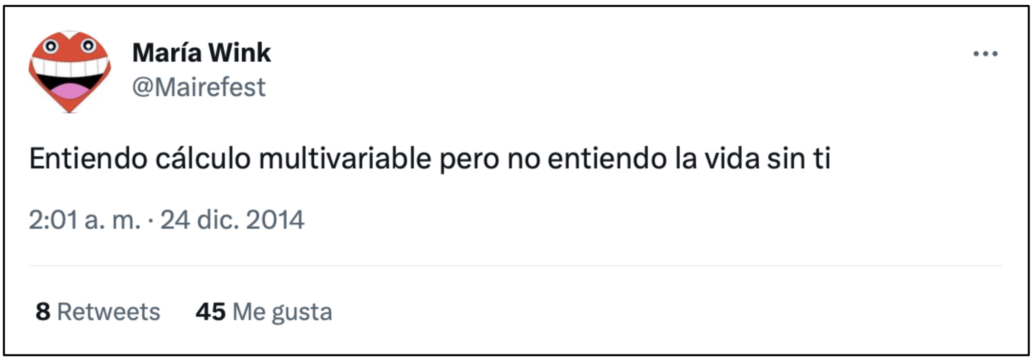


**Figure 5: Tweet by @Mairefest retrieved from https://twitter.com/Mairefest/status/547663369923473408**

This article section aimed to illustrate how social networks can serve as a medium to gain insight into students' emotional experiences while studying mathematics in school. This kind of data source can enhance our comprehension of how students perceive and engage with mathematics in an educational setting.

## Using the internet as a source of mathematical help

School mathematics is an educational experience where students commonly encounter doubts. It is also common for students to seek sources of help to clarify these doubts. The sources of mathematical help that students resort to can be varied: their classmates, the mathematics teacher, a family member, or a book, among others. Thus, help-seeking is an intrinsic part of studying and learning school mathematics. The help-seeking behaviors that enable students to clarify their doubts independently can be interpreted as manifestations of self-regulated learning.

Along with Danelly Susana Esparza Puga from the Universidad Autónoma de Ciudad Juárez in Mexico, we have studied how digital resources such as the internet, mobile devices, and social networks shape the help-seeking behaviors of mathematics students. Our initial explorations focused on internet-based mathematical help-seeking practices among Mexican engineering students (Aguilar & Esparza Puga, 2015; Esparza Puga & Aguilar, 2015). Specifically, we sought to answer the following questions:

- What websites do students consult when they need help in mathematics?
- What do students use those websites for?
- Why do students trust the mathematical information provided by such websites?

The research method to address these questions primarily focused on self-reports from participating students provided through focus groups and individual interviews. The findings showed that the most frequently used sites for mathematical help-seeking were the Google search engine, Facebook, and YouTube. The latter was identified as the most popular source of help among the participating students. Regarding the uses that students give to these sites, the following were identified:

- Finding different ways to solve a mathematical problem
- Clarifying doubts and reinforce knowledge
- Getting ready-made results or mathematical problems solved
- Comparing their results or answers to problems with other answers to similar problems found on the internet
- Catching up with a class they skipped

Regarding the issue of trust in mathematical information provided by these websites, we began to notice that students do not seem to pay attention to the intrinsic mathematical properties (Lithner, 2003) of the obtained information but rather base their assessment on features not related to mathematics, such as the academic prestige of the person or institution that publishes information. These two student statements illustrate this situation:

Student: YouTube seems reliable to me because university teachers upload the videos.

Student: SlideShare...I think is more reliable because there the doctors [PhDs] send [slides presentations].

## Delving into the use of YouTube as a source of mathematical help

The outcomes of initial exploratory studies prompted us to delve deeper into the type of mathematical assistance students search for on sites like YouTube and the reliability criteria they use to validate the mathematical information they discover there. We pursued this research by making methodological improvements to complement the self-reports and interviews used in the exploratory studies. In particular, we employed surveillance software to record students' activity when seeking mathematical help online (see methodological details in Aguilar & Esparza Puga, 2020). Direct observation of students' online activity when attempting to solve a mathematical problem allowed us to identify two types of help-seeking behaviors among students.

The first type is *executive help-seeking,* which refers to situations where students aim to find something or someone to assist them in solving a problem or achieving a goal on their behalf. For example, when students turned to community-driven question-and-answer websites like "Yahoo! Answers," where they could effortlessly obtain answers to specific mathematical tasks. This type of behavior, in which students seek mathematical help in community forums—some of them without getting too involved in the construction of the answer or solution—has been previously reported by van de Sande (2011).

The second type of help-seeking behavior identified was *instrumental help-seeking,* where students' searches are more focused on promoting a self-understanding of an idea or a problem-solving process. For example, we found evidence of a student who identified a YouTube video through a Google search based on keywords such as "definite integral exercises solved," "calculate area under a curve," and "area under the quadratic equation curve." The student could extrapolate the integration technique they learned by repeatedly watching the YouTube video to solve a mathematical task involving solving the integral $\int_0^4 (-x^2 + 4x)\, dx$ (see Aguilar & Esparza Puga, 2020).

In the study by Esparza Puga and Aguilar (2023), the general characteristics of the mathematical help students obtain through YouTube videos are explored, particularly regarding the qualities of the sources they prefer and trust. Using a popular channel of videos on school mathematics called "julioprofe" as a reference (see http://youtube.com/julioprofe), we interviewed first-year engineering students who used these videos. The interviews aimed to identify (1) the characteristics of the mathematical help that students obtain through these videos, and (2) the criteria for reliability that students use to trust—or not trust—the mathematical information obtained from these sources.

The results reveal general characteristics of the mathematical help students obtain through this type of video:

- *It is multifunctional*. Through these videos, students get multipurpose mathematical help. They can use it when they have attended class but have doubts and want to clarify them, or they can use it to introduce themselves to a new mathematical topic. They can use it when they cannot attend class and want to catch up on lessons. In addition, the mathematical help they get from these videos could cover different school mathematical topics, from the most basic to the most advanced.
- *It is always available*. Another prominent feature of this mathematical help is that it is available anytime, anywhere—as long as the student has internet access. Students can turn to

this source of mathematical help in and out of school and, as one student put it, “if you are doing the assignment at 3 in the morning, it [the help] is always there.”

- *It is private*. Students can refer to this source of help privately, without revealing their doubts to their classmates or the lecturer. Some studies suggest that students may feel ‘dumb’ in front of their peers when asking for help or expressing doubts in mathematics class (Newman & Schwager, 1993). The mathematical help obtained on YouTube eliminates these inconveniences because it can be consumed privately.
- *It is easy to use and self-paced*. Some of the interviewed students highlighted the brevity and simplicity of the videos by julioprofe, as well as the presenter’s step-by-step explanations. In addition, the students have personal control over the pace since the video can be stopped, skipped, or repeated as many times as needed.

As for the criteria students use to trust the mathematical information contained in these videos, the findings suggest that trustworthiness is based on three elements:

- *People close to them recommend it*. Several students describe how their lecturers, parents, or classmates recommended that they look for mathematical help on YouTube or the julioprofe channel. We think that the fact that authority figures such as their lecturers or their parents recommend it—in addition to their classmates—promotes students’ trust in this source.
- *It works*. Another element that we believe increases students’ confidence in this source of mathematical help is that it has helped them to solve assignments and even pass exams—as some of the students interviewed report. We think that when students receive positive notes and evaluations after using julioprofe’s videos to study, they interpret it as tangible proof of the effectiveness of those videos as study support.
- *It gets ‘likes’ and positive comments*. Students pay close attention to the ‘likes’ and comments that the videos receive from other YouTube users. Some interviewed students analyze the number of ‘likes’ and the kinds of comments a video receives to weigh its quality. In the case of julioprofe, the videos receive thousands of ‘likes’ and positive comments.

## Concluding discussion

In this paper, two points have been illustrated. Firstly, social media serves as a space where individuals express their emotions related to school mathematics. These online social spaces can be used as a window into people’s attitudes and feelings about school mathematics. Researchers and educators can better understand how individuals feel about school mathematics and how they engage with it by analyzing public sentiment on social media. This approach can lead to developing effective strategies and interventions to improve attitudes towards mathematics and enhance learning outcomes. The use of Twitter as a tool for monitoring and analyzing public sentiment toward school mathematics has the potential to advance our understanding of this important subject area.

Secondly, this paper has highlighted how the internet and social media have changed how students search for mathematical help and validate mathematical knowledge. We are witnessing a shared epistemology among new generations of students, in which mathematical knowledge is independently obtained beyond the walls of the mathematics classroom. Its certainty or truth is validated not based on its intrinsic mathematical qualities but through indicators of the authority of the sources and other social indicators such as recommendations, comments, or the number of likes obtained by the source of mathematical information. This transformation in acquiring mathematical knowledge and validating its certainty has significant implications for mathematics education and calls for new approaches to teaching and learning mathematics that consider the role of social media and internet resources.

The emergence of a new epistemology in which mathematical knowledge is validated through indicators of authority on social media, such as likes, comments, and recommendations, is a significant shift in how students perceive the value and reliability of mathematical information. This new way of validating knowledge transforms the traditional notion of mathematical authority and expertise. Students increasingly look beyond traditional sources of authority, such as teachers and textbooks, to validate their understanding of mathematical concepts. In this new epistemology, the trustworthiness of mathematical knowledge is based on the collective judgment of a community of users on social media platforms, who provide feedback on the quality and relevance of the information shared. This has implications for how we understand the nature of mathematical knowledge and the role of authority and expertise in the field of mathematics. Furthermore, it highlights the importance of digital literacies in mathematics education. Students must learn how to evaluate and critically assess the credibility of mathematical information found on social media platforms.

While this new epistemology challenges traditional approaches to teaching and learning mathematics, it also offers opportunities for innovation and collaboration in the field. By embracing the power of social media and digital technologies, educators can engage students in new and exciting ways, facilitating meaningful and authentic learning experiences that align with the changing nature of mathematical knowledge in the digital age. Therefore, mathematics educators need to recognize and address the emergence of this new epistemology, developing pedagogies that encourage critical reflection and evaluation of mathematical information found on social media platforms while also promoting a deeper understanding of the nature of mathematical knowledge and the role of authority and expertise in the field.

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## Section 2

# CONTEXTS AND PERSPECTIVES: COMPETENCIES, IDENTITY, AND INTERDISCIPLINARITY

# The relation between skills and competencies in the KOM framework of mathematical competencies: A discussion

Oliver Kauffmann[1] and Uffe Thomas Jankvist[1]

*In this paper, we explore an influential conceptualisation of mathematical competencies, the KOM-framework (Niss & Jensen, 2002; Niss & Højgaard, 2011, 2019). In particular, we take a critical look at the relation between competency and skills. This specific discussion reflects a more general concern: Why and how should we care about the body in discussions of mathematical competencies?*

*Keywords: Competencies, skills, supervenience, embodiment.*

## Introduction

One way to address the questions about the body's role for mathematical competencies is to adopt a metaphysical perspective. Competencies of a mathematical kind in human beings, whatever a proper analysis will reveal about their nature, are all—at a minimum—contingently related to the body. Let us cautiously begin our discussion by taking a look at this idea.

Assume for the sake of argument that the mathematician and philosopher René Descartes was right, when he found about our epistemic capabilities of mathematics and geometry that knowledge of simple entities like numbers and figures is possible by intuition and deduction only (Descartes, 1966). Since intuition and deduction, according to Descartes, take place in the soul, which, according to his argumentation, is a substance (i.e., an entity the existence of which does not depend on anything else) different from the substance of extended bodies, the body is not, apparently, important for these capabilities at all. However, even if Descartes was right in his epistemic assumption about the soul's prominent role for mathematical and geometrical insights, these activities of the soul would still be related to the body, although in a contingent, causal way. We indulge for instance in thinking about the possible solution to a mathematical problem. We try out different solutions; we act, reach out for paper and pencil, jot down numbers and equations by tapping our keyboard, scratch our scalps while we ponder, move our eyes, move in the chair and so on and so forth. Causal interactions take place between our thoughts and our body. Thus, even when a Cartesian view on metaphysics is assumed, i.e., a conception according to which two substances exist—the material one extended in space, and the thinking substance not extended in space—the body and the thinking soul stand in a causal relation to each other. The explanatory details of this assumed interaction are highly debatable, as it has been since Descartes' days. When we are doing a mathematical calculation, applying our relevant skills, knowledge, and competencies, this is only the instantiation of one out of many possible subsets of this causal story. Substance dualism has been extensively criticized and is, if not fallen completely out of favor, at least considered as a last resort in philosophy of mind (for a good introductory account pro et con substance dualism, see Kim, 2010; for a recent defense of substance dualism, see Foster, 1989). On the other hand, a number of materialist positions of a reductive sort advanced through the

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second half of the $20^{th}$ century, have also met critique. They have been found wanting to the extent that they do not explain a number of peculiar characteristics of the mind, in particular the qualitative aspects of conscious awareness and the feature of some or all mental states, that they exhibit intentionality (i.e., have representational content). Instead, many philosophers and scientists have turned to so-called 'non-reductive accounts' of the relation between the body and the mind. In our discussion of the role of the body in relation to mathematical competencies, we also have a non-reductive account of the mind as our background assumption. To be a bit more precise, we endorse a version of 'the supervenience thesis,' about the relation between body and mind. This condition is also in line with some thoughts behind the KOM-framework of mathematical competencies and serves as a thinking tool in this domain.

## The supervenience thesis

The psychophysical supervenience thesis is an instance of nonreductive materialism. In more general terms, a set of M properties ('M' for 'mental') is said to supervene on a set of P properties ('P' for 'physical') with respect to a domain D just when two entities in D, which happen to be indiscernible with respect to P, are necessarily indistinguishable with respect to M. That is to say, two entities in D that differ with respect to M necessarily differ with respect to P, but not conversely. Thus, the possibility of the existence of entities, which differ with respect to P but not with respect to M, is not excluded. In other words, this is an instance of an asymmetrical dependency relation between a set of (so-called) supervening properties M and the(ir) 'supervenience base,' the set of relevant properties P. If D is the domain of psychophysical relations, the thesis of psychophysical supervenience says that all psychological or mental properties M (states, event, processes) supervene on the physical properties P (we use the terms 'psychological' and 'mental' interchangeably.) All mental properties are dependent on the bodily properties, and not the other way around (cf. e.g., Kim, 1982). The rationale for psychophysical supervenience is threefold. Firstly, this assumption accommodates that the psychological sphere has properties, which are not properly accountable for in a reductive explanatory framework. Thus, psychophysical supervenience leaves room for mental properties 'over and above' the physical domain per se; these properties are different from the physical (including bodily) properties, although they depend on the physical properties. In particular, interest has been shown in dealing with two specific mental features within a supervenience framework: intentionality and consciousness. 'Intentionality' refers to the specific, relational feature of all (or most) mental phenomena. To be in a mental state involves an object of that state: what the state is about. Mental states are directed at something. They represent something. We cannot think, desire, know or entertain a belief without thinking of, desiring, knowing or believing something. For short: mental states represent. 'Consciousness' refers to the particular way some living organisms are capable of representing. When an organism is conscious of something, there is a particular way for the organism to be in that mental state. To be consciously aware of something means that 'there is something it is like for the organism to be in that state' (Nagel 1974, p.436). For a zombie, in contrast, there is not something it is like for it to be in its representational state. A supervenience account is a nonreductive account of the mental to the extent that it acknowledges the peculiarities of intentionality and consciousness. Here it differs from blunt reductive theories like behaviorism and materialism. On the other hand, the asymmetric dependency-relation between the mental and the physical in psychophysical supervenience still gives the physical domain priority. This feature is what makes the position palatable for many with a naturalistic leaning. Secondly, psychophysical supervenience apparently leaves room for the existence of mental-to-mental causation and mental-to-physical causation: My desire to find the solution to a mathematical problem makes me think, and makes me grab for a pencil, etc. As Kim has pointed out, however, the psychophysical supervenience thesis might run into severe troubles with both mental-to-mental and mental-to-physical causation (Kim, 1998, 2005). Yet, we cannot go into these delicate matters here. Thirdly, and most important, psychophysical supervenience catches the intuition, that the mental, despite being something 'over- and above' the physical, is still dependent on the physical, not only by not being a substance of its own, but also by being dependent on the physical.

What this 'dependency' precisely comes to is the crux of the matter in the domains where the interrelations of knowledge, skills, and competencies are discussed—and, as we shall see, in the discussions of embodiment. In accordance with psychophysical supervenience, the physical is primary, and the mental is secondary to the physical. We do have empirical evidence for the mental properties being (somehow) causally dependent on the physical (in particular from the neurological and the neuropsychological domains), and do not have evidence for the physical as supervening on the mental. Out of the vast quantity of 'physical particulars' (particles and fields in spacetime) we are acquainted with, only a very small subset exhibits mental properties, whereas (apparently) all known mental properties are connected with particulars of a physical kind.

## Competencies depend on skills and knowledge

The background condition about supervenience in the psychophysical domain is relevant for dealing with the connections between competencies, skills, and knowledge in education. The idea is simple. Competencies (somehow) *depend on* skills (and knowledge), *without being reducible* to such. And just as mental properties from the perspective of psychophysical supervenience are not conceived as free floating, but instead depend on the existence of embodied organisms, neither do competencies exist as free floating 'properties', but are described as features or actions of embodied human beings with certain skills and knowledge. Since achievement of skills depend on the body, and the competencies depend on the achieved skills and knowledge, the competencies *somehow* depend, at least partially, on the body. The competencies partly rely on our bodies to the extent that competencies (partly) rely on skills, and skills only come into the world through our learning bodies. This being said, various points of view can be adopted, for instance for taxonomical or evaluative purposes, from which competencies *per se* can be described and explored. This is similar to the possibility of exploring structural features of the mind *per se*, such as intentionality and conscious awareness, as mentioned.

## Mathematical competencies

The notion of mathematical competencies, as opposed to mathematical skills and knowledge, has gained momentum within the past decades in mathematics programmes, not only in Scandinavia and Northern Europe, but also for example in Columbia and not least in the international assessments PISA (OECD, 2019). Kilpatrick states that school mathematics sometimes "is portrayed as a simple contest between knowledge and skill" while "Competency frameworks are designed to demonstrate to the user that learning mathematics is more than acquiring an array of facts and that doing mathematics is more than carrying out well-rehearsed procedures" (2014, p. 87). And certainly, competency has become a key construct today in the educational paradigm within various areas (cf. Sadler, 2013; Stacey, 2010; Stacey & Turner, 2014), overshadowing and replacing former, dominant constructs such as knowledge and skill. As examples of competency frameworks, Kilpatrick (2014) mentions three: (1) the five strands of mathematical proficiency as identified by the Mathematics Learning Study of the US National Research Council; (2) the five components of mathematical problem-solving ability identified in the Singapore mathematics framework, and (3) the eight competencies of the Danish KOM framework. The Danish KOM framework (Niss & Jensen, 2002; Niss & Højgaard, 2011) was also implemented as the basis of the PISA framework of mathematical competencies (e.g., Stacey & Turner, 2014). More precisely, seven of KOM's eight mathematical competencies were part of PISA until approximately 2018. In addition, there is now also the Chinese Core Mathematics Competencies framework (MOE, 2018, 2022).

Either explicitly or implicitly, all these competency frameworks will have to come around an explanation of the relations between competencies, skills and knowledge. The authors behind the KOM framework deal explicitly with these relations, and by applying the supervenience thesis into this discussion along the lines indicated, we are able to pinpoint and discuss the essential 'hinge conceptions' of the relation between procedural skills, knowledge, and competencies and suggest how to potentially develop the KOM framework further along this line of thought.

## The KOM framework and supervenience

The basic, original idea of KOM was to formulate the concepts of mathematical competence and competencies "with particular regard to their possible roles in the teaching and learning of mathematics" (Niss & Højgaard, 2019, p. 10). Hence, the main thought behind the KOM-framework was that the teaching of mathematics would be able to promote the students' development of these competencies and the related kinds of 'overview and judgment.' The overall characterization of 'mathematical competence' in accordance with KOM is described as "having knowledge of, understanding, doing, using and having an opinion about mathematics and mathematical activity in a variety of contexts where mathematics plays or can play a role" (Niss & Højgaard, 2011, p. 49). This overarching conception of competence spans eight distinct, yet mutually related, separate competencies, often illustrated by an 8-leafed flower, the so-called KOM-flower as shown in Figure 1.

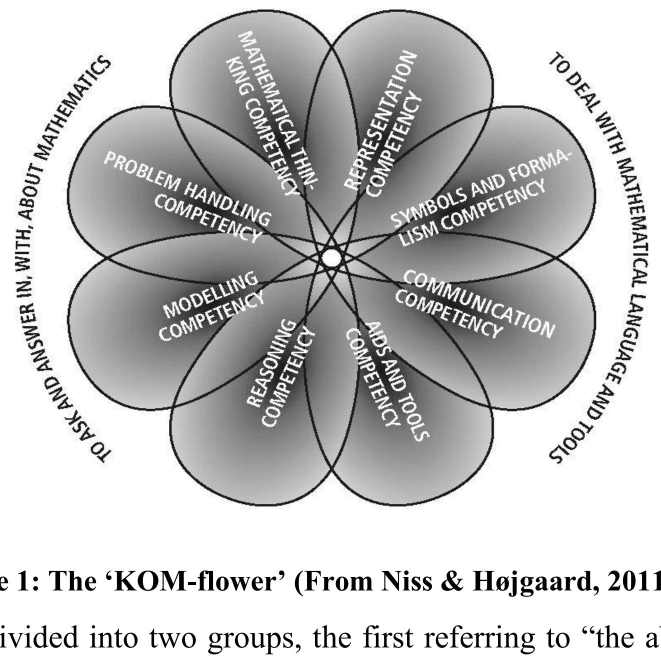


**Figure 1: The 'KOM-flower' (From Niss & Højgaard, 2011, p. 1)**

The competencies are divided into two groups, the first referring to "the ability to ask and answer questions in mathematics and with mathematics" (Niss & Højgaard, 2011, p. 50); the second to "the ability to deal with mathematical language and tools" (p. 50). The first group covers the four competencies of mathematical thinking, problem handling, modelling and reasoning. The second group covers the competencies of mathematical representation, symbols and formalism, communication and aids and tools. Both of these groups have (almost exclusively) explicitness (through language) as characteristics, and generally the competencies are conceived as cognitive of nature (cf. e.g., Niss & Højgaard, 2019, p. 12). It is important to notice that none of the eight competencies can be possessed and developed in complete isolation from other competencies; hence, the non-empty intersection at the centre of the KOM-flower (Figure 1).

Recently the authors of the original KOM-report found reasons to revisit the conceptualization of the basic notions "in order to provide an updated version of the original conceptual framework and terminology" (Niss & Højgaard, 2019, p. 9). In this paper, they give a concise, but instructive exposition of the specific relation between knowledge, procedural skills, and mathematical competencies. But why *procedural* skills? In the very detailed 2011-report (Niss & Højgaard, 2011), procedural skills are not addressed at all. In contrast with the generic term 'skills,' the term 'procedural' only appears once (in the expression "procedural knowledge" (p. 49)). In the 2019-paper, however, 'procedural skills' are in focus. This probably reflects the upsurge of interest in, and acknowledgment of, the importance of action for understanding mathematical competencies. Competence is now considered as "someone's insightful readiness to act appropriately in response to challenges of given situations" (Niss & Højgaard, 2019, p. 12). With reference to what was already recognized by George Pólya (1957/1945), they see enactment of mathematics "as the essential constituent in the mastery of mathematics" (2019, pp.12–13). Let us therefore take a look at procedural skills versus competencies.

By 'procedural skill in mathematics,' the authors mean a person's ability to perform, with accuracy and certainty, a particular, methodologically well-defined—oftentimes algorithmic—goal-oriented type of undertaking (Niss & Højgaard, 2019). What they suggest is that the relation between procedural skills and competencies can be conceived as a specific type of compositional relation. The existence of competencies relies on the existence of procedural skills, but without the competencies being reducible to these skills (Niss & Højgaard, 2019). They also claim that skills can be seen as necessary, but not sufficient for the competencies. `The explanations for competencies not being reducible to skills, and for the claim that skills are not sufficient for competencies, must probably, at least partially, be found in the following condition: "exercising a given competency typically requires the activation of a multitude, probably hundreds, of different, very specific procedural skills, each of which draws upon a reservoir of factual knowledge. For example, the symbols and formalism competency involve the procedural skill of performing rule-based transformations of algebraic expressions in different mathematical domains or determining the derivatives of combinations of standard functions" (p. 20). Let us dub this condition, 'the many-one constraint.' Regarding the authors' idea about a specific kind of compositional relation, they elaborate with a metaphor from chemistry. Huge molecules (e.g., polymers), composed out of atoms, result in the existence of new properties, and competencies are comparable with the properties of such molecules, the atoms of which (with their lower-level properties) are the procedural skills (Niss & Højgaard, 2019). Although this conception in the domain of chemistry and biology is perhaps better known as 'emergentism,' it is not different from the supervenience thesis (the authors do not use these theoretical labels), and psychophysical supervenience can consequently be seen as a special case of emergentism (cf. e.g., Kim, 2010). With a definition of emergentism from Kim: “When aggregates of material particles attain an appropriate level of structural complexity (‘relatedness’), genuinely novel properties emerge to characterize these systems” (Kim, 2006, p. 292).

How does this particular ‘picture’ of the relation between skills and competencies advanced by Niss and Højgaard fare? Clearly, the higher-order properties themselves, differentiated as they are in the eight different leaves of the KOM-flower, are applicable in the preparation of specific curricula, for the evaluation of performances, and in other ‘praxis directions.’ The authors end up their 2019 paper by listing a number of ‘educational uses,’ and certainly the KOM-framework has a strong track-record of applicability. The theoretical underpinnings of KOM are also helpful for developing a deeper understanding of skills and competencies. Clearly, the competencies rely on the skills, in the sense that the competencies do not come into existence without the skills, and the picture of emergence is perhaps truly helpful here. It gives us a hand in uncovering a bit more about what this ‘reliance’ would amount to. Part of the idea with emergentism is that the bottom layer of the particles *necessitates* the existence of the aggregated, more complex layer of the system with its new properties. The bottom layer plays an aggrega*ting* role for bringing the aggrega*ted* level of (new) properties into existence. This means, that an instantiation of a particular set of aggregating properties necessarily *leads to* the set of aggregated properties. Mentioned *en passant* with respect to this specific point, emergentism and supervenience may come apart. Thus, it is debated whether supervenience (as defined above in this paper) also entails this stronger ‘necessitating feature’ between the base level and the supervening level of properties. (cf. e.g., Kim, 2010, p. 9). Yet, with respect to Niss and Højgaard (2019), this necessitation from the base level is a stronger claim than merely saying, as they do, that the base with properties P is necessary, but not sufficient for M, or, in terms of skills and competencies, that skills are necessary, but not sufficient for the competencies coming into existence. In accordance with emergentism, the aggregating properties, i.e., the supervenience base, is sufficient for bringing about the emerging properties. Transferred to the domain of skills and competencies, the skills would then after all be sufficient for bringing competencies into existence. This is obviously not what Niss and Højgaard claim, but if the metaphor with emerging properties is taken at face value, this is what their view implies, and it therefore appears to harbour an inconsistency.

Instead of giving in on Niss and Højgaard’s assumption that procedural skills are necessary, but not sufficient for competencies, we should likely give in with respect to the usefulness of the model of emergentism instead. The problematic part of emergentism applied to skills and competencies is the condition about ‘necessitation.’ Competencies are certainly related to various procedural skills, through the subject’s acquisition of them—but the competencies do not come fully into existence from the mere acquisition of these skills. Remember that they are not reducible to the skills themselves, according to Niss and Højgaard. As mentioned above, part of the explanation for this probably lies in ‘the many-one constraint.’ If we give in with respect to emergentism because of a too close bond between skills and competencies (via ‘necessitation’), how should we then understand the relation between skills and competencies?

Perhaps we could say—slightly vaguely—that competencies only ‘bloom’ through a person’s *applying* the acquired skills and knowledge in various *contexts*. This would be in line with the last part of Niss and Højgaard’s definition: “Competence is someone’s insightful readiness to act appropriately in response to the challenges of given situations.” (Niss & Højgaard, 2019, p. 12). Would it be clearer to seek a Solomonic middle ground by preserving that competencies, on the one hand, (really) are acquired through learned skills (and knowledge) and are attributable as properties of persons (p. 11), but, on the other hand, only (really) come into (mature?) existence through application in shifting (new?) contexts? This would be a way to accommodate our willingness to speak of types of levels in mastery of a specific competency, e.g., in dealing with symbols and formalism or with mathematical modelling (cf., Niss & Højgaard, 2019, pp. 21–22). Still, it can be asked, are competencies as properties of persons attributable to subjects apart from their being applied? Put differently, can persons possess them *independently* of their attribution? The acknowledgement of the importance of this aspect of competencies is clearly indicated by the first part of Niss and Højgaard's definition: the 'readiness to act.' This line of argumentation would make competencies look more like so called 'dispositional properties' (like solubility, flammability, etc.). Perhaps then a 'readiness to act' account of mathematical competencies, where these are not reducible to learned skills (+knowledge), fares better?

If we ignore the attribution-aspect, and instead seek an account of the skill-competency relation in terms of 'readiness to act,' how would such an account look, then, if we remind ourselves, that 'competencies cannot be reduced to skills (+knowledge)'? Well, why are competencies considered not to be reducible to (relevant) skills? The central elements of the supervenience thesis plus what we dubbed 'the many-one constraint' is helpful with an answer to this. Remember that supervenience is an asymmetrical dependency relation. Competencies need skills (+knowledge) in order to exist—not the other way round. On the other hand, competencies are different from the skills they rely on. In the scholarly discussions of the nature of mind and body, it is obvious what the specific candidates for 'nonreductive' properties are, motivating for theories accommodative for mental properties being different from physical properties—despite the former being dependent on the latter for their existence. Consciousness and intentionality are such 'top candidates,' as we have seen in the motivation for psychophysical supervenience. How do competencies in any comparable way 'stand out' from skills from the perspective of Niss and Højgaard? What we called 'the many-one constraint' describes how a given competency typically requires the activation of a multitude, probably hundreds, of different, very specific procedural skills, each of which draws upon a reservoir of factual knowledge" (Niss & Højgaard, 2019, p. 20). How does the quantitative difference between 'the many skills' and 'the one competence' add up to a qualitative difference of a nonreductive sort? Why can't we just say that a specific competency is nothing more than the sum of the individual elements (skills) it relies on, and therefore reducible to those skills (+the individual sets of knowledge related to each skill)?

If (and only if) competencies are considered *in abstraction* from the contexts of application, uses, exercises, and challenges, they will be reducible to the constituent skills. Each competency would *arguendo* then just be the mere instantiation of the arithmetic sum of its constituent parts, the relevant skills. The competencies thus considered are mere resultant, additive properties of their parts. So, in abstraction, we might after all be able to consider a competency as a dispositional property of a person. Yet, in the concrete, things look differently. "The core of mathematical competency," Niss and Højgaard declare, "is the enactment of mathematics in contexts and situations that present a certain kind of challenge" (2019, p. 20). No such challenges arise from the abstract perspective of competencies as 'readiness to act' on one's acquired knowledge and skills. The potential irreducible character of a competency comes from the transfer of an individual's acquired knowledge and skills into a new situation. Competencies from this 'conjunctive perspective' are from an abstract perspective reducible to sets of skills, the learning of which leads to 'readiness to act' in a dispositional sense. Yet, they are irreducible precisely when they are challenged through being put to use in new contexts. Still, the question of the precise role of the body within a framework like this calls for further reflections. A number of mental features such as our capabilities for abstract thinking, modelling, and representing equations and numbers, continue to challenge truly embodied accounts of mathematical competencies.

## Embodiment and mathematical competencies

In the last three-four decades, we have been witnessing a turn in cognitive sciences, social sciences, and the humanities: the turn toward understanding learning, feeling, and cognizing as situational enacted, embedded, extended, and embodied activities, standardly referred to as '4E' (Lakoff & Johnson, 1999; Shapiro, 2019; Varela et al., 1991). An additional number of Es such as 'Emotion' have been added over the years, but only very recently, 'Education' has arrived. These conceptualizations of embodied cognition have also had an impact on the understanding and exploration of learning and skills within mathematics (see e.g., Lakoff & Nuñez, 2000). A number of 'alternative,' 'implicit' skilled ways to learn various mathematical concepts, such as 'proportional equivalence' (e.g., Abrahamson et al., 2021; Hutto et al., 2015), have challenged cognitivism by claiming that these skills are not requiring any appeal to explicit, content-full representations, but instead appear to be based on "content-free enactive explorations that unfold within learning contexts" (Hutto et al., p. 385). The literature on embodied mathematical cognition gives suggestions to the effect that mathematical skills from a learning perspective ultimately rely on bodily skills (Lakoff & Nuñez, 2000; Nuñez et al., 1999). In (Hutto et al., 2015), for example, proportional equivalence (e.g., instantiated by '6:10 = 9:x') was learned by students through developing a bimanual motor-action scheme, where the manipulation of handles dynamically correlated with visual feedback from cursors on a screen enabled the students to learn 'proportional equivalence' without the involvement of any mathematical symbolism. Ideas like these invite for pursuing reconceptualizations of procedural knowledge in mathematics (e.g., Star, 2005), and give evidence to the effect that certain mathematical tasks can be accomplished by alternative, embodied, enacted procedures (e.g., Abrahamson et al., 2021; Donovan & Alibali, 2021). Still, one central set of issues, that continues to tease and frustrate researchers who address skill-based perceptual or cognitive task in attempts to find non-cognitive 'alternative' ways to accomplish them, is the plethora of high-level tasks of cognition, such as thinking and having reflective conscious awareness. These issues certainly appear pertinent and relevant to address for educational purposes, where the emphasis on formal ways of teaching and didactical thinking has been the tradition.

Clearly, the KOM framework is conceived from a cognitive perspective. So much the more surprising it is that Niss and Højgaard (2011) at the same time are very much aware of a number of elements (potentially) closely connected to an embodied perspective, which they leave out or behind: "we have maintained mathematical competence and competencies as basically cognitive constructs. In so doing, the significance of affective, dispositional and volitional factors of mathematical mastery and learning has in no way been disregarded, but these factors are of a different nature to the ones taken into account in this framework" (Niss & Højgaard, 2019, p. 26). Also, their very definition of competence as "someone's insightful readiness to act appropriately in response to the challenges of given situations" (2019, p. 12) should be discussed in the context of so called 'enacted theory,' a theory complex in cognitive science of a 4E kind, which emphasizes the importance of cognitive agents' implicitly knowing the interrelations between their sensory inputs and motor capabilities (see e.g., Hutto (2005) for some varieties of enacted theory).

Even if we do not have evidence for the strong claim that all types of mental states supervene directly on the skilled habitual behaviour of agents, the existence of alternative, embodied routes to obtaining mathematical tasks of various sorts indicates the bandwidth and plasticity of our learning repertoire and demonstrates that importance of this research for educational purposes.

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# Computational thinking and mathematics viewed as interdisciplinarity

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*The international trend towards young people learning programming and computational thinking in compulsory school has led to different developments in mathematics teaching. In this paper, we view this as a case of interdisciplinarity and explore what the interplay between mathematics and computational thinking in K-9 schools in Denmark, Sweden and England looks like through this lens. Based on national curriculum standards, resources and knowledge about educational practices, we compare intentions and realities in these three countries. The analysis suggests that the school systems in the three countries all look at the interplay between mathematics and programming as interdisciplinarity—but as different types of interdisciplinarity. In England, we see a weak integration of disciplines driven by a pragmatic need, in Sweden a deeper integration of disciplines, and in Denmark there is a focus on understanding and overview across disciplines.*

*Keywords: Programming and computational thinking, technology, mathematics, interdisciplinarity.*

## Introduction

Across the globe, there is a general push in the direction that young people should be taught programming and computational thinking in compulsory school (Bocconi et al., 2022). In some countries, this ambition is addressed through mathematics teaching (Tamborg et al., 2023). This makes it interesting to view the interplay between technology education and mathematics education as a case of interdisciplinarity in itself, where, in the best of all cases, there can be a mutual fertilization between disciplinary and pedagogical traditions. From a historical perspective, we observe a movement from an early (beginning around 1980) interest in children programming to improve their mathematical capabilities and provide new ways of working with mathematics, to a period with a lesser focus on the interplay between programming and mathematics (roughly 1990-2010) (Clements & Sarama, 1997). We now find ourselves in the middle of what can be characterized as a “second wave” of the interplay between programming and mathematics education (Lodi &

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Martini 2021). A little before 2010, people from computer science and industry stated that children need to learn about computing—or "computational thinking" (Wing, 2006). In the years to follow, some countries began to take up this challenge and implemented computer science and programming in compulsory school.

This paper explores the questions: *To what extent may the interplay between mathematics and computational thinking in K-9 schools in Denmark, Sweden and England, be viewed as examples of interdisciplinarity? And, how does this unfold on the level of national curriculum standards, in resources and in educational practices?*

The main purpose of addressing these questions is to create an "orienting framework" for the navigation of resources to support the teaching of programming and computational thinking and mathematics. The questions are addressed by a theoretical discussion viewing examples of practices, resources and political documents from the three countries Denmark, Sweden and England. These three countries all work with implementing programming, computational thinking, digital competencies, and technology as part of their curricula. Hence, by juxtaposing the approaches we might be able to see more clearly the nature of the interdisciplinary relations between computational thinking and mathematics. In the paper, we first describe the research project "Programming and Computational Thinking and Mathematical Digital Competencies," and the theoretical and methodological constructs that we apply.

## The project "Programming and Computational Thinking and Mathematical Digital Competencies"

In the project, "Programming and Computational Thinking and Mathematical Digital Competencies," we take as outset that we are in the middle of an international wave that pushes implementation and Programming and Computational Thinking (PCT). With that outset, we look at the policies and practices in the three different countries of Denmark, Sweden and England. We apply the Danish competencies framework (Niss & Højgaard, 2011; 2019) augmented with the notion of "Mathematical Digital Competencies" (MDC) (Geraniou & Jankvist, 2019) in order to study and compare the way PCT is implemented. MDC involves awareness of which tools to apply for what mathematical purposes, using digital technology reflectively for problem-solving and learning, and engaging in a techno-mathematical discourse.

Within the project, we have combined a comparative study of how PCT is integrated with mathematics teaching at the compulsory level in the three countries with design experiments. To develop synergies between mathematics teaching and PCT in the Danish school system, we have compared approaches and resources from the three countries (this work is described in Elicer & Tamborg, 2022; 2023). Based on these findings, we have conducted design-based interventions to verify and refine our understanding of potential synergies between PCT and mathematics, and we are currently in a process of developing easy-to-use teaching sequences as an educational resource for all teachers (Misfeldt et al., 2022).

## Theory and method: Comparing interdisciplinarity

We view the way that computational thinking is incorporated in the mathematics curriculum through the lens of interdisciplinary configurations. The idea is that this lens will allow us to look at the way that mathematics meets PCT in teaching and in curricular documents.

Jensen and Jankvist (2018) distinguish three kinds of interdisciplinarity addressing different types of needs.

(1) Interdisciplinarity as *integration of disciplines on the border of two existing disciplines.* Examples are mathematical economics, bioinformatics and geophysics as examples. This kind of interdisciplinarity is *driven by a demand for specialization*.

(2) Interdisciplinarity as the *integration of disciplines*. Examples are business studies, which integrates economics, sociology, law, etc.; medical science, which integrates chemistry, physiology, psychology, etc.; engineering, which integrates mathematics, physics, geology, etc. This type of interdisciplinarity is driven by a demand to integrate elements from basic disciplines to create an applied discipline.

(3) Interdisciplinarity as *understanding and overview across disciplines*, or as a kind of “Allgemeinbildung,” a type of interdisciplinarity that is a response to negative and unfortunate consequences of disciplinary specialization. This involves creating the need for bridge-building and the ability to see problems through different disciplinary lenses.

We will use these three types of interdisciplinarity to look at the Danish, the Swedish and the English responses to increased focus on programming and technology education, more specifically how mathematics and PCT are related in school discourse and practice in the three countries.

Using a combination of an inductive and a deductive comparative strategy, we show the differences and similarities in the configurations of interplay between PCT and mathematics (Bereday, 1967). More precisely, we shall use the three kinds of interdisciplinarity developed by Jensen and Jankvist (2018) to develop a decontextualized comparison of the relationship between the teaching of PCT and that of mathematics in the three countries. We view the implementation stories from the three countries as critical cases, where the everyday mathematics teaching acts to support the comparison. In the comparison, we build on the short implementation stories described below. These descriptions are reconstructions based on previous research that we refer to in the individual sections describing the approaches of the three countries.

## Implementing programming and computational thinking in the three countries

In England, one main reason for focusing on computational thinking is to support the industry (Tamborg, 2022). The structure resembles the scientific discipline. In a sense, it is a kids’ version of computer science, which has been implemented in the educational system since 2013 as the school subject “computing” (Misfeldt et al., 2020). In Sweden, the work with programming and computational thinking has been much closer related to the subject of mathematics. In 2018, it was decided that all Swedish students should work with programming in relation to their mathematics classes, from grade 1 through grade 12 (Bråting & Kilhamn, 2021). At compulsory school level, programming was integrated in close connection to algebra and at upper secondary level mainly in relation to problem solving. Furthermore, this initiative was motivated by equity, and especially by an aim to avoid a new type of digital divide between people who can create value with technology and people who increasingly become consumers and spectators to life through technology (Heintz et al., 2017). Denmark, on the other hand, has been experimenting with a new topic of “technology comprehension” in a number of schools for the past four years. Technology comprehension has been tested as both a subject in its own right, and as a part of other existing school subjects, e.g., first language (L1), mathematics, science, and social science. In both cases, the motivation has mainly been citizenship and democratic empowerment (Smith et al., 2020). The focus for technology comprehension encompasses design competencies, computational thinking, citizenship, and technical skills.

We now provide a more detailed account for each of the three countries, beginning with Sweden.

## Details about the Swedish approach

In 2017, the Swedish K-9 curriculum was revised with more focus on students' *digital competence*. This led to the revision of all major school-subjects, and in this process, programming became part of mathematics throughout grade levels 1 to 9 (Heintz et al., 2017; Swedish National Agency of Education, 2018). The Swedish mathematics curriculum is organized in six areas: Understanding and use of numbers; Algebra; Geometry; Probability and statistics; Relationship and change; and Problem solving. Programming was added to algebra at all grade levels and described in the following way (Swedish National Agency of Education, 2018, pp. 56–59):

- Grades 1–3: How unambiguous, step-by-step instructions can be constructed, described, and followed as a basis for programming. The use of symbols in step-by-step instructions.
- Grades 4–6: How algorithms can be created and used in programming. Programming in visual programming environments.
- Grades 7–9: How algorithms can be created and used in programming. Programming in a visual and text-based programming environment.

For grades 7–9, algorithms are also mentioned in relation to problem solving, and computer simulations are mentioned in relation to statistics:

- How algorithms can be created, tested, and improved in programming for mathematical problem solving.
- Assessment of risk and chance based on computer simulations and statistical material.

Computational thinking is not mentioned in the Swedish national curriculum document. Instead, the focus is mainly on algorithms and programming. The Swedish curriculum displays a clear progression. In the early years, programming is approached by focusing on step-by-step instruction and symbols, often in so-called unplugged activities. In middle school, focus is on algorithms in visual environments, and from grade 7 using text-based programming (Bråting & Kilhamn, 2021). Helenius and Misfeldt (2021) analyze the Swedish and Danish situation and noticed that programming itself is the main focus in Sweden, whereas there is less focus on how programming can be used as mathematical tools. Another characteristic of programming in the Swedish mathematics curriculum is its focus on practices and problems, whereas programming concepts and data structures, are more or less absent.

## Details about the Danish approach

Denmark has not yet made a final decision on revising the curriculum to include informatics-related topics (as of 2023). Nevertheless, in 2018, the Ministry of Education initiated a pilot project where 46 schools began teaching a new subject called technology comprehension with the ambition to teach K-9 students the ability to critically relate to and shape technology (Børne- og Undervisningsministeriet (BUVM), 2018). The project began with developing a curriculum for technology comprehension as a subject in its own right. This curriculum included four competence areas, namely: Digital empowerment; Digital design and design processes; Computational thinking; and Technological agency. Each competence area was defined by three to five subject matter areas presented as pairs of skill set and knowledge. In the case of computational thinking, the subordinated subject matter areas are: Data; Algorithms; Structures; and Modeling.

As previously mentioned, the project experimented with two different implementation strategies; (1) as an independent subject as described above and (2) as integrated into other subjects, among those, mathematics. Six technology comprehension components were integrated in mathematics: (a) digital design and design processes; (b) modeling; (c) programming; (d) data, algorithms and structures; (e) user studies and redesign; (f) computer systems. (BUVM, 2019). Even though these elements are added to the curriculum, they are not explicitly related to existing mathematical competencies and subject matter areas. This results in a curriculum consisting of juxtaposed components from technology comprehension and mathematics, thereby placing the responsibility of developing meaningful integrations in concrete teaching on the shoulders of teachers.

## Details about the English approach

England was among the very first countries to focus on PCT in relation to compulsory school. This happened in 2014 with a new computing curriculum (Department for Education, 2014a). The computing curriculum replaced a prior mandatory Information and Communication Technology (ICT) curriculum. This topic mostly covered basic usage of technology and information search and the ability to assess quality of information (Department for Education and Employment, 1999). A central reason for this change was that the ICT curriculum fell short in preparing students to contribute to the ICT industry, which was important for economic growth in England. Furthermore, there was a decline in recruitment in ICT and computing courses (Council of Professors and Heads of Computing, 2009; the Royal Society, 2012; Microsoft, 2007).

The argument that schooling should promote PCT since industry needs ICT professionals is not unique to England, but the English case is very clear in how this motivation leads to policy and implementation. The English computing curriculum can be seen as a simplified version of computer science as taught in tertiary education. In headlines, this curriculum aims at:

1. understanding and applying fundamental principles of computer science,
2. the ability to analyze problems in computational terms,
3. evaluating new and familiar technologies, and
4. to educate students to become responsible, competent, confident and creative users of technology. (Department for Education, 2013)

The content consists of: (a) algorithms and programming; (b) logic; (c) computational thinking; (d) digital content and potential uses for digital technology; (e) safety and citizenship; and (f) systems, search and software (Department of Education, 2013). The teachers of the previous ICT subject were assigned the responsibility of teaching computing (Larke, 2019). This, despite the fact that "ICT" and "computing" were highly different. Formally, there is no relation to mathematics, but both teachers of mathematics and mathematics education researchers have taken up the focus on programming and computational thinking (e.g., Benton et al., 2017; 2018). Hence, on the level of pedagogical practice the relation between mathematics and PCT does exist.

## PCT and mathematics teaching in the borderland between school subjects and political intentions with schooling

We view the problem of understanding the interdisciplinary configuration between programming and computational thinking, on the one side, and mathematics, on the other, on a continuum from policy intentions and documents, over resources to actual teaching practice.

We have previously argued (Tamborg et al., 2022) that the politically decided curricular structures that specify the interplay between programming and mathematics can be described as displayed in figure 1. Sweden has implemented deep integration. This is visionary, but it also naturalizes the relation between programming and mathematics. In Sweden, programming is focused on algorithms and it is closely related to algebra.

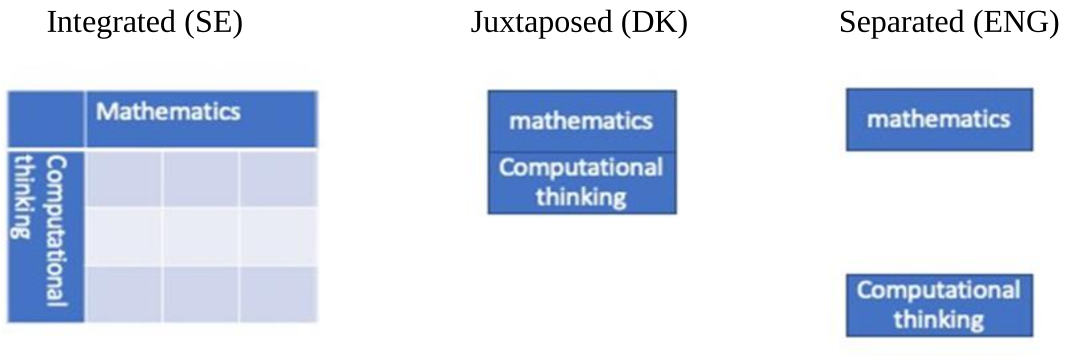


**Figure 1: Three types of curricular integration between mathematics and computation thinking. From Tamborg et al. (2023)**

In Denmark, the details of the relation between programming and mathematics were not specified in the curriculum, but the intention of relating mathematics and technology is clear. In the English case there is no relation between mathematics and programming, but both feed into a pragmatic concern about raising students' abilities in relation to technical and mathematical situations.

In order to compare the policy level intentions and official curricula in more detail, we can distinguish scholarly knowledge and practical knowledge (Helenius & Misfeldt 2021). On the level of scholarly knowledge, the Danish approach is broad, since it combines computer science, sociology of technology and experiments with integration into several of the school subjects (mathematics, science, first language and craftsmanship). England has a focus on the academic discipline of computer science and on educating for becoming part of the technology industry. In Sweden, the strong relation to mathematics, and especially to algebra, means that the scholarly knowledge addressed is focused on the intersection of computer science and algebra, in particular algorithms and variables. Regarding the practical knowledge addressed in the curricular documents, the Danish technology comprehension has an affinity to design thinking and the ability to prototype and develop technology. Furthermore, the Danish approach puts emphasis on analyzing the influence of technology on society. The practical knowledge, which is the focus of the English and Swedish documents, is much more related to programming and algorithms. The differences between the scholarly and practical knowledge in focus are shown in Table 1.

**Table 1: The way that the curriculum documents focus on scholarly knowledge and practical knowledge**

| Country | Scholarly knowledge | Practical knowledge |
|---|---|---|
| Denmark | Broad knowledge of technology, and technology as part of topic/in topic | Design of technology and analysis of technological situations |
| England | Computer science for kids in a sense monodisciplinary | Programming and technology development |
| Sweden | Mathematics, algebra and algorithms, viewed as an integrated topic | Algorithms and programing addressing mathematical problems |

If we look at the teaching resources that are provided for working with PCT, we can also see a clear difference in the focus and way that the interplay between technology and computing is addressed. Based on examples, we have previously shown that the materials in the three countries differ (Elicer & Tamborg 2022; Misfeldt et al., 2020). We have analyzed teaching resources from Denmark and Sweden to discuss the difference in how they address this interdisciplinarity (Elicer & Tamborg, 2022; Misfeldt et al., 2020). Because of the status of the curricula, there are published textbooks in Sweden with programming integrated into mathematics. In Denmark, these resources are online teaching units for the pilot project on technology comprehension. Some resources focus mostly on programming and algorithms while others take a broader view, one including the handling of data and computer modeling. Danish tasks cover a wide range of mathematical topics—arithmetical, geometrical, and statistical—concepts, while Swedish tasks mainly focus on patterns and sequences.

As for the English case, there is no such intent of making PCT and mathematics an interdisciplinary field. However, there are resources produced and made available by, for example, the *ScratchMaths* project (Benton et al., 2017). Their core contribution is a collection of teaching resources that express what they can do for students' mathematical learning. This is the so-called 5E's pedagogical framework for action: Explore; Envisage; Explain; Exchange; and bridgE. The latter attempts to have students connect computational and mathematical ideas.

Our comparable data about the teaching practice in the three countries consist of expert interviews (further elaborated in Tamborg et al., 2022). Based on this, we see that in Denmark there is a broad covering of all aspects of PCT in comparison with England and Sweden. However, while we in the public discourse see a relatively clear orientation in Denmark towards democratic education, this is less clear when looking at the combination of interviews and curriculum (Tamborg et al., 2022).

In England, it was a surprise that socioeconomic inequality was so present in the analysis. Previous research (Tamborg et al., 2022) unanimously found a clear orientation in English policy toward building a strong competitive workforce.

## Discussion: Interdisciplinarity as a window to national differences when implementing PCT

The notion of interdisciplinarity helps us understand some of the differences between how PCT is implemented in Denmark, Sweden and England.

In England, we see what Jensen and Jankvist (2018) describe as interdisciplinarity as *integration of disciplines on the border of two existing disciplines, driven by a demand for specialization*. On the policy level, we see more or less no integration between mathematics and PCT, as the *computing topic* is formally not attached to the mathematics curriculum. Still, the purpose of adding computing to the curriculum is not completely disjoint from the larger purpose. Even though the computing curriculum as such is unrelated to the mathematics curriculum, the English case shows that this does not imply that no integration is pursued. Firstly, integration can be left to teachers and material developers as in the *ScratchMaths* project. Furthermore, the basic incentive of the curriculum has been to develop a strong workforce that can specialize in technology development. In a sense, the interdisciplinary interplay between mathematics and PCT in the English school system is not driven by overt political intentions. The unfolding of the interdisciplinary relation between mathematics and PCT is rather driven by non-policy agents, such as teachers, scholars and NGOs. Nevertheless, there is a rich environment oriented towards experimenting with making use of PCT as a part of mathematics education. In addition, mathematics and computing are aligned on the level of values, since the overall political intention of implementing PCT is to enhance students' inclination towards careers in technology development. This leads us to suggest that the English situation is framed by a demand for specialization and the need to develop strong skills at the intersection between mathematics and PCT. That is, it is interdisciplinarity of the first type since the tech entrepreneur integrates elements from both mathematics and computer science to develop new solutions in a specialized borderland between the two disciplines. Even though the goal of integrating mathematics and PCT is relevant and to some extent articulated in relation to the English implementation of PCT, there is no such thing as an attempt to develop a new topic or integrated scholastic discipline. The need for integration is pragmatic rather than epistemic.

In Sweden, we rather see what Jensen and Jankvist (2018) refer to as interdisciplinarity as the *integration of disciplines*. Jensen and Jankvist provide examples of *business studies* that integrate economics, sociology, law, etc.; *medical science*, integrating chemistry, physiology, psychology, etc.; and *engineering*, which integrate mathematics, physics, geology, etc. This type of interdisciplinarity is driven by a demand to integrate elements from basic disciplines to create an applied discipline, in Swedish case one of algebra and algorithms with an independent object and an epistemic approach.

On the curriculum level, we have entitled this as an *integrated* approach to PCT and mathematics. PCT components are integrated into specific mathematical areas (e.g., programming in algebra). While this on the one hand offers little autonomy for teachers, it does on the other hand attempt to create a meaningful and more complete disciplinary package of objects and methods that allows students and teachers to move beyond a pragmatic need for specialized skills and towards a more fully integrated disciplinary identity. This is seen in the ambitions and structure of the national curriculum standards as well as in the tendency to address genuine mathematical problems with PCT in the Swedish materials and resources (Elicer & Tamborg, 2022). Nevertheless, there are also indications that the integration is not complete. Bråting and Kilhamn (2021) show that there is still a way to go before the programming tasks that are being promoted always serves a mathematical goal, and Helenius and Misfeldt (2021) show that the integration between mathematics and PCT is still mainly on the practical level and less developed on the theoretical level.

In the Danish case, there is a strong resemblance to what Jensen and Jankvist (2018) describe as *understanding and overview across disciplines*. This is a form of general education or "Allgemeinbildung" (Bildung). As previously mentioned, this type of interdisciplinarity is a response to negative and unfortunate consequences of disciplinary specialization and involves creating the need for bridge-building and the ability to see problems through different disciplinary lenses. The ambition of looking at technology as digital empowerment, digital design and PCT, and from various disciplinary perspectives, including—but not limited to—mathematics, does reflect an ambition on a very high level of abstraction. The specific description on how mathematics and technology can be related in the Danish curriculum are, nonetheless, more juxtaposed than integrated, since the details in the relation between PCT and mathematics are largely left to the teachers. Nevertheless, there is a number of indications that the scaffolding is in fact too weak for the current state of implementation. Hence, there is a risk that this autonomy does not really lead to a realization of interdisciplinarity as understanding and overview. Helenius and Misfeldt (2021) contrasted the Swedish—"practice without theory"—integration between mathematics and PCT, to a Danish situation of "theory without practice." Research on Danish teachers' usage of teaching materials based in relation to the experimental program in technology comprehension shows that the materials are followed with high fidelity, which could indicate that teachers do not feel skilled to adapt the material on their own (Børne- og Undervisningsministeriet, 2021). In that sense, Denmark clearly aims at a Bildung style of interdisciplinarity, i.e., that of type three. Still, the currently weak teacher capacity and slow approach to implementation challenges this ambition, since the teachers tend to only follow predetermined materials. We present an overview of the various approaches and levels in Table 2.

**Table 2: Interdisciplinary configuration at various levels**

| Country | Policy level | Resource level | Teaching practice and expert level | Configuration |
|---|---|---|---|---|
| England | Separated, but shares values of driving recruitment for technology development | Programming and technology development is addressed by numerous resources | Structured by schools/colleagues. Heavy use of resources. | Interdisciplinarity type 1- focusing on specialization into technology professions |
| Sweden | Integrated | Algorithms and mathematical problem solving | Structured by the national level. Tendency to practical integration without theoretical integration | Interdisciplinarity type 2 – developing an applied discipline of algebra-algorithms and computing |
| Denmark | Juxtaposed | Mission, design and empowerment driven | Autonomy in principle - but strong dependence on materials in practice. | Interdisciplinarity type 3: understanding and overview across the many aspects of digitalization |

## Conclusion: PCT and mathematics as two of many things

In this paper, we have shown that the integration of PCT into school can relate in a number of different ways to mathematics teaching. One way of looking at these different approaches is through the lens of interdisciplinarity. This lens has allowed us to untangle some of the differences in ambitions and reality of England's, Sweden's and Denmark's ways of relating PCT and mathematics teaching as different ways of doing interdisciplinarity. The question of if the interplay between mathematics and computational thinking in K-9 schools in Denmark, Sweden and England, may be viewed as examples of interdisciplinarity can be answered with a 'yes'. Still, we can also see that there are ways in which the ambitions of the curricular documents are not completely fulfilled in the developed resources and in educational practices. Especially Sweden seems to attempt to develop a sort of applied discipline between algorithm and algebra, but in the current practice there are little theoretical concerns of why this makes sense and what it can offer. Similarly, Denmark seems to be attempting at developing an approach to teaching technology that highlights understanding and overview across various fields, but lacks practical foundations in technology development, programming and computational thinking.

The fact that we can view the interplay between PCT, on the one hand, and mathematics, on the other, as very different cases of interdisciplinarity in the different countries calls for further research. We have previously tried to understand the necessity and nature of mathematical digital competences (MDC) (Geraniou & Jankvist, 2019) as an attempt to update the normative aspect of mathematics instruction to meet digitalization. Nevertheless, the division of labor between mathematics and PCT is of a different nature in the three countries under scrutiny, meaning that there is a further need to understand what parts of MDC that reside "naturally" in mathematics teaching and what parts that may potentially be outsourced to a computing curriculum.

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# The Lifestyles Project: An interdisciplinary integration of mathematics with science, arts and English in the Québec context

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*Learning mathematics in a silo presents it as a complex discipline that is detached from the real world. As such, there is a growing interest in interdisciplinary mathematics education (IdME) in Québec's secondary education. However, novice secondary teachers in Québec find it challenging to adopt IdME due to a dearth of literature providing concrete examples of this approach. Thus, this narrative practice-based paper discusses a descriptive example from a Québec school where IdME took place in a Secondary 2 (Grade 8) classroom through the "Lifestyles Project." The Lifestyles Project was sub-divided into three consecutive assignments (Hobbies, Careers, and Bedroom Design), interspersed in the mathematics course throughout the year. In each of these assignments, students were graded using rubrics designed by the two collaborating teachers of the integrated subjects. Learning from the example of the Lifestyles Project, we encourage researchers and educators alike to further examine the ways to develop IdME initiatives and transform mathematics instruction to make it engaging, meaningful, and relevant for the students.*

*Keywords: Interdisciplinary mathematics education, Québec secondary education, project-based learning*

## Introduction

Mathematics knowledge and processes are used in multiple aspects of everyday life. Engaging in mathematics fosters critical thinking and problem-solving skills, which are also crucial to addressing complex global issues. Although mathematics plays an important role in society, most students still find it vague, abstract, and detached from reality (Mosvold, 2008). As a result, interdisciplinary mathematics education (IdME) has received considerable attention in recent years (Chao-Fernández et al., 2019; Chi, 2021; Kokko et al., 2015). In line with this trend, Québec's Education Program (QEP) emphasizes the importance of interdisciplinarity and cross-curricular competencies for meaningful and engaging mathematics learning (Québec Ministry of Education, 2007a, 2007c, 2007d). As underscored in the QEP, "teachers should encourage students to discover the connections that may be made with other subjects…through interdisciplinary activities in the classroom or the school" (Québec Ministry of Education, 2007c, p. 14). Students can improve their mathematical understanding if they are able to connect it with other fields of knowledge/disciplines (Québec Ministry of Education, 2007c).

Despite the QEP's emphasis on connecting mathematics with other disciplines, interdisciplinarity remains an ongoing concern in Québec secondary education. Novice secondary teachers face multiple

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challenges in adopting interdisciplinary teaching and learning into their practices (Hasni et al., 2015; Lenoir & Hasni, 2010). Particularly, there is a dearth of literature that provides concrete examples and evidence of IdME to learn from (Lindvig & Ulriksen, 2019). In an attempt to address the highlighted issue, our narrative practice-based paper discusses an example from a Québec school where IdME took place in a Secondary 2 (Grade 8) classroom. In this paper, we aim to provide a detailed description of the implementation of an interdisciplinary project-based learning approach in a real-life setting. This interdisciplinary mathematics project will serve as an example of interdisciplinary learning and teaching within secondary mathematics education.

## Curriculum integration models and IdME

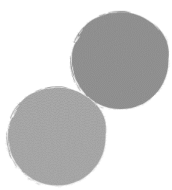
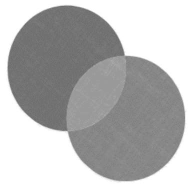
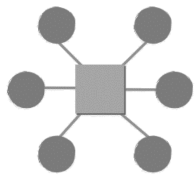
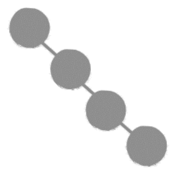
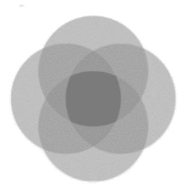
The notion of integrated curriculum implies that different forms of knowledge and disciplines will work together to impart meaningful learning (Pring, 1971). In an integrated approach to teaching and learning, multiple disciplines are fused and combined together instead of teaching them separately (Costley, 2015). There are a variety of models of curriculum integration that exist to help practitioners understand and implement curriculum integration in their classrooms.

## Models of curriculum integration

One model for curriculum integration (Vars, 1991) offers teachers three options for combining different disciplines, namely, correlation, fusion, and core curriculum. ‘Correlation’ is the simplest, most basic form of integration, where different subject teachers combine two or more disciplines in one topic simultaneously. Correlation is used to reveal connections between disciplines within a single subject area. For example, the mathematical concept of geometric figures might require discussions about strength of structures and symmetry, drawn from the subject of science. Taking correlation a step further, Vars (1991) describes ‘fusion’ as an approach that involves merging the content of two or more disciplines to create an entirely new discipline. For example, a unit of study about water may be designed by fusing concepts from mathematics, science, and geography. The most advanced and complex approach in Vars’ model is the ‘core curriculum,’ which takes on a student-centered approach to curriculum integration. In this approach, the integrated curriculum’s nature and criteria are determined based on the needs and problems of the students. Teachers group students’ needs and problems into clusters and then design integrated study/content units that integrate skills and subject matter content from any pertinent subject, suited to the original problem/need. As a result of this approach, the criteria for integration are to determine what subjects and forms of knowledge are essential to explore and solve the problem at hand.

In contrast to Vars’ model, Fogarty’s (1991) model proposes five approaches to integrate curriculum across disciplines: sequenced, shared, webbed, threaded, and integrated. Somewhat similar to Vars’ ‘correlation’, the ‘sequenced’ approach involves teaching relevant topics from different disciplines independently. These topics are sequenced to provide a framework for broad concepts and to facilitate the transfer of learning across content areas. The ‘shared’ approach focuses on the shared concepts, ideas, and skills as knowledge of two disciplines connected through a common topic. These common topics promote shared instructional experiences among learners, allowing them to make connections across the two integrated disciplines. Moving beyond two disciplines, the ‘webbed’ approach employs a thematic approach where three or more independently taught disciplines are combined such that they lean toward a common theme as a base for instruction. This approach helps learners to make sense of the topic under study by drawing knowledge from multiple disciplines. As the name suggests, the ‘threaded’ approach involves integrating the disciplines together through the common thread of a chosen skill. In this approach, the content of the integrated disciplines is only a tool for learners’ skill development, such as social, thinking, technology, and study skills. Like the ‘core curriculum’ approach by Vars, Fogarty’s threaded approach is student-centered. However, in contrast to students’ needs and problems, the nature and criteria of curriculum integration in the threaded approach depends on the selected students’ skills. In this approach, the combination of disciplines is established by what skills the students need to develop. Lastly, in the ‘integrated’ approach, a common goal or theme, which requires knowledge of more than two disciplines, is proposed. In this approach, the overlapping aspects of the included disciplines encourage learners to see the interconnectedness and interrelationships between the disciplines. Graphical representations of each approach in Fogarty’s model are given in Table 1.

**Table 1: Fogarty’s model of curriculum integration across disciplines, adapted from Fogarty (1991)**

| ***Approach*** | **Sequenced** | **Shared** | **Webbed** | **Threaded** | **Integrated** |
|---|---|---|---|---|---|
| ***Representation*** |  |  |  |  |  |

Another model of curriculum integration was later presented by Drake & Burns (2004), which greatly overlaps with some approaches of Fogarty (1991) and Vars (1991), see Figure 1. According to this model, teachers can integrate curriculum in three ways, starting from a ‘multidisciplinary’ approach, which is at the lowest level of integration, to ‘interdisciplinary’ and finally a ‘transdisciplinary’ approach, which is at the highest level of integration. Similar to the correlation and webbed approaches of Vars and Fogarty respectively, the approach of multidisciplinary integration involves organizing the standards from multiple disciplines. These disciplines are combined such that there exist interconnections between the disciplines despite teaching them separately. In contrast to the multidisciplinary approach, in the ‘interdisciplinary’ approach, teachers emphasize such concepts and skills that are common to two or more disciplines. Thus, the interdisciplinary approach resembles the ‘integrated’ approach of Fogarty’s model as both involve combining disciplines based on their common characteristics. Irrespective of the disciplinary and/or interdisciplinary skills and concepts, the ‘transdisciplinary’ approach focuses on the development of innovative solutions to real life problems by using concepts and skills of all the disciplines. This approach greatly resembles Vars’ core curriculum approach as both take the ‘problem-centered’ approach as a basis of combining disciplines.

## Interdisciplinary mathematics education

IdME occurs when an ‘interdisciplinary’ approach to curriculum integration is used to teach mathematics (Chao-Fernández et al., 2019). In IdME, mathematics is often combined with science, arts, and/or languages (Lovemore et al., 2021; Serrano Corkin et al., 2020). The interdisciplinary integration of mathematics presents mathematics in a wider context (Chi, 2021); combined with real-world knowledge to foster problem solving skills and inquiry among students (Williams et al., 2016).

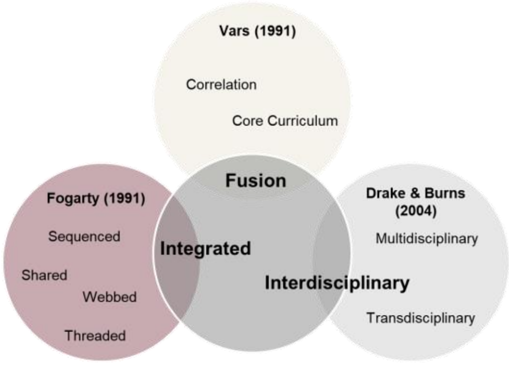



**Figure 1: Venn diagram showing relations and interconnections between Fogarty (1991), Vars (1991) and Drake & Burns (2004) models of curriculum integration**

IdME also develops students' understanding and sense of how mathematics relates to, and builds on the other subjects (Kokko et al., 2015). When students discover these interconnections between mathematics and other subjects, their beliefs and attitudes towards mathematics are positively transformed (An et al., 2008). As many students find mathematics vague, abstract, and detached from reality and real-life situations (Mosvold, 2008), IdME underscores mathematics as an engaging discipline, which is inherently present in other disciplines as well as everyday knowledge (Chi, 2021). Given the many benefits of IdME, most teachers have positive perceptions towards using interdisciplinary approach of curriculum integration in their classrooms (Ozturk & Erden, 2011; Saleh & Shaker, 2021).

## Project-based learning within IdME

There are multiple approaches that can be used by teachers to implement IdME in the classroom. A common and widely used approach for IdME is project-based learning (PBL) (Chi, 2021). PBL is an instructional method that allows students to engage meaningfully with interdisciplinary content at a deeper level (Condliffe, 2017). In PBL, students investigate challenging questions or issues of personal interest in authentic and real-life settings (Aydın-Günbatar, 2020; Bell, 2010; Laur, 2013; Lin et al., 2015; Virtue et al., 2019). When students apply mathematics knowledge to real-life situations, their mathematical understanding is greatly improved (Boaler, 1997; Holmes & Hwang, 2016). PBL helps students in "establishing conceptual associations between the learnt knowledge" (DemiRel & Coskun, 2010, p. 48) and hence, improves their academic achievement in mathematics (Gürgil & Çetin, 2018). Indeed, a study by Holmes and Hwang (2016) found that students who initially had little belief in their mathematics skills "expressed mastery or learning goals" as a result of engaging with PBL (p. 459). Students "show stronger motivation than in relation to more traditional didactic models" (Ghisla et al., 2010, p. 10) when learning mathematics in an interdisciplinary, project-based format.

## The Lifestyles Project: Conception and design

In this section, we present an example of an IdME project, which took place in a Secondary 2 mathematics classroom at a Québec school.

## Project background

The IdME project was developed within a context of a research-practice partnership with a small, independent girls' school in Québec, Canada. The Grade 8 mathematics teacher in the school, Stephanie (all names are pseudonyms), was keen to implement the interdisciplinary approach (Drake & Burns, 2004), particularly IdME, into her mathematics course using PBL as a means to achieving this goal. As such, she developed the "Lifestyles Project," which was designed in collaboration with teachers from other subject areas (science, English language arts (ELA), and visual arts).

The overarching objective of this project was to implement IdME as a curriculum integration approach such that it would:

a) engage students in mathematics education by discovering its connections with other disciplines and everyday knowledge (An et al., 2016; Chi, 2021);
b) align with the Québec's Secondary 2 mathematics curriculum (Québec Ministry of Education, 2007d); and
c) help students develop the skills aligned in Québec's Broad Areas of Learning (Québec Ministry of Education, 2007c) and Cross-Curricular Competencies (Québec Ministry of Education, 2007b).

## Project schedule & set-up

To achieve the project objectives, the Lifestyles Project was sub-divided into three consecutive assignments: the Hobbies, Careers and, Bedroom Design assignments, see Table 2. The project schedule was set up such that all assignments were purposefully interdisciplinary and interspersed in the mathematics course throughout the year, each spanning 2–3 weeks in length.

## Hobby assignment

The first assignment of the Lifestyles Project was the Hobby assignment, which was comprised of four classroom sessions and involved the integration of the mathematics and science courses. Stephanie (the mathematics teacher and project lead) and Melinda (the science teacher) collaborated to design the Hobby assignment. The goal of the Hobby assignment was for students to explore at least one science and one mathematics concept from their chosen hobby and then share their work.

***Session 1:*** In the first session, with the support of Melinda, Stephanie introduced the Hobby assignment description and grading rubric to the students for their review. Then, they facilitated a whole-class discussion where students collectively defined and explained what constitutes a 'hobby' while discussing various examples of hobbies based on their real-life experiences and interactions.

After establishing a base knowledge of what constitutes a hobby, each student selected a hobby of personal interest to explore and research the mathematics and science inherent in the activity. Students chose a wide range of hobbies, such as sewing, swimming, horseback riding and cooking.

**Table 2: Summary of Lifestyles Project assignments**

| | **Hobby Assignment** | **Careers Assignment** | **Bedroom Design Assignment** |
|---|---|---|---|
| # Classroom Sessions | 4 | 8 | 5 |
| Integrated Subjects | Mathematics & Science | Mathematics & English Language Arts (ELA) | Mathematics & Arts |
| Collaborating Teachers | Stephanie and Melinda | Stephanie and Melissa | Stephanie and Penny |
| Goal | Explore at least 1 science and 1 mathematics concepts in the chosen hobby and share the final work | Choose a potential career and calculate the net and gross annual salary considering the tax deductions | Design a scaled, 3D model of the bedroom of your dream apartment while taking into consideration certain constraints |
| Key Topics | - Definition and description of hobby<br>- Process of conducting and referencing an authentic and credible online research<br>- Discovering science and mathematics in a hobby | - Mining potential of aptitude tests for careers choice<br>- To search and interpret online information related to job, salary and tax deductions of the careers<br>- Analysis of a pay statement based on given salary scale<br>- Calculating the pay deductions and bi-weekly, monthly, and annual gross and net salary | - Designing a blueprint or floor plan of any room<br>- Graphing scaled top-view drawings<br>- Calculation of a total surface area of any 3D design<br>- Use of low-relief cardboard technique and decoration using patterned paper<br>- Designing based on elements and principles of art |

***Session 2:*** In the next session, Melinda discussed the topic of conducting an online authentic and credible research as students had limited exposure of, and experience with the online research. In this discussion, Melinda facilitated the dialogue on ‘authenticity and credibility’ of online information and citation of online sources, see Figure 2a. This discussion provided students an opportunity to both develop their research skills as well as prepare for the research component of the Hobby assignment. Then, students started working on their individual assignments in the last half of this session (Figure 2b) as Stephanie and Melinda facilitated them through their individual work of online research and responded to their queries.


**Figure 2: a) Melinda leading a discussion on credibility of online sources in the first half of session; b) students doing their individual work on the Hobby assignment with support from Melinda and Stephanie**

***Session 3:*** In the third session of the Hobby assignment, students continued to work on their individual assignments. Stephanie and Melinda reviewed and restated assignment expectations and rubrics to support students' success.

***Session 4:*** In the last session, students submitted their work in multi-modal formats including write-ups, video essays, and presentations, see Figure 3. The students were required to share: 1) at least one science and one mathematics concept from their hobby; 2) the rationale behind why they chose a hobby; and 3) the list of references they used to carry out the assignment. Some students also opted to share their final product with the whole class.

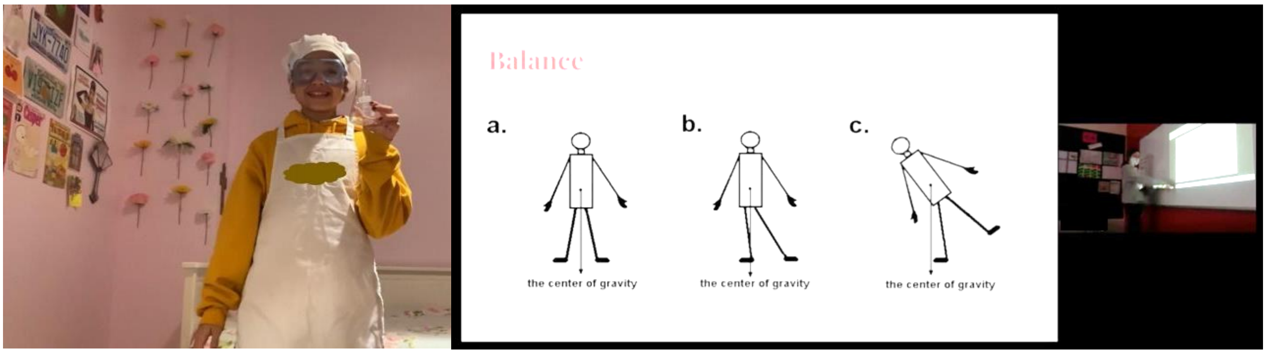



**Figure 3: a) Screenshot from the video submitted by Secondary 2 student Raza presenting the scientific aspects of cooking; b) Cymbi presenting the science and mathematics concepts in ballet**

## Careers assignment

The second assignment of the Lifestyles Project was the Careers assignment. For this assignment, Stephanie collaborated with Melissa (the ELA teacher). This assignment was specifically designed so that the students can go beyond their hobby and start thinking about translating their interests into potential careers. The scope of this assignment was much broader and involved eight classroom sessions, i.e., twice as many classroom sessions as the Hobby assignment.

***Session 1:*** In the first session, students took aptitude tests, shared their top results on a digital interactive whiteboard (Jamboard) and selected their careers. Students could either pick a career from the results of their aptitude test or choose a different career that matched their interests and hobbies.

***Session 2:*** During the second session, students were required to find three jobs within Canada based on their career choice. Using a template designed collaboratively by Stephanie and Melissa, students added the job details to a 'job notesheet'. These details included the job description and qualifications, reasons the students were interested in these three jobs and APA citations for the job sources. To promote peer collaboration among students, students were asked to post helpful websites on a Jamboard, see Figure 4.


**Figure 4: Screenshot of the websites shared by students to find jobs in Canada**

***Session 3:*** During the third session of the Careers assignment, students continued to research jobs and complete the job notesheet. The focus of this session was to help students correctly complete the notesheet while adding APA citation properly using BibMe, an online bibliography generator.

***Session 4:*** While working on their individual job notesheet, see Figure 5, students engaged in multiple reflection discussions in this session of the Careers assignment. In these discussions, students talked about the process of looking for a job, their personal job preferences, interests, and lifestyles. The students continued to work on their individual assignments while engaging in these whole-class discussions.

| | |
|---|---|
| What qualifications do you need for the job? Ex: education? prior experience? | DEC or Bachelor Degree in Fashion Design 1 to 2 years experience in technical sketches Strong retail background and trend awareness Knowledge of garment construction and fittings Detail-oriented Strong verbal and written communication skills Solid computer skills, Illustrator and Photoshop Excellent team cooperation Excellent product knowledge Good sense of organization Bilingual, French and English, written and spoken. |
| Does the job description provide a starting salary? If so, what is it? | 29k per year. |
| What is the job description? What responsibilities or tasks does the job ask for? | Create precise technical drawings Enter all fabric and spec information Build detailed tech packs utilizing designer's initial flat sketch & trim sheet |
| Would you be interested in this job? Why? | yes because I'd be designing and seeing how things work. |

**Figure 5: Screenshot from Lassa's job notesheet**

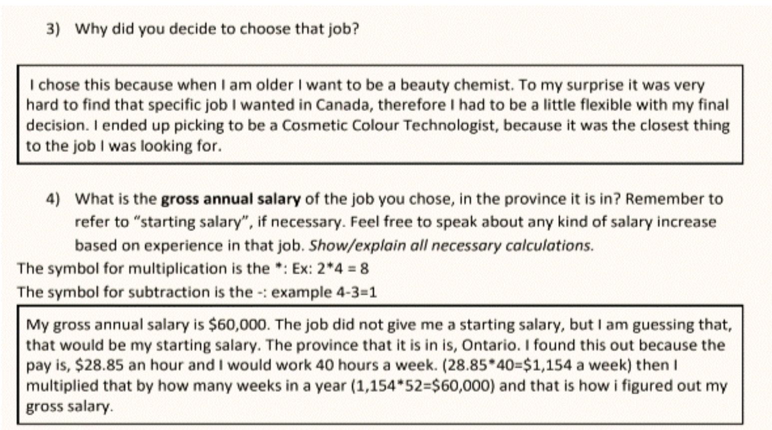
***Session 5:*** The fifth session of Careers assignment started with a 'Mr. Needy salary activity'. In this activity, students had to analyze a pay statement by using a real-life context of a Québec teacher, referred to as Mr. Needy. To complete this activity, students used Classkick, an online software, which allowed students to get help from their peers and teachers as they needed, see Figure 6. Students were given the salary scale of a general Québec teacher as well as the pay statement of Mr. Needy and asked to interpret this information. Next, students answered questions regarding pay deductions and bi-weekly, monthly, and annual salary calculations. Lastly, students researched different terms associated with salary and tax deductions, such as “QPIP” (Québec parental insurance plan) and “QST” (Québec sales tax) and identify if these deductions are common to jobs in Québec, Canada and particularly Mr. Needy’s salary statement. First, students did mathematics calculations, followed by research and writing (using templates developed in collaboration with ELA) in the second half. Besides individual work, students participated in whole-class discussions about gross versus net salaries, different pay deductions, and how they differ from province to province in Canada.

**Figure 6: a) Stephanie facilitating the students while they are completing Mr. Needy assignment on Classkick; b) group of Secondary 2 students working individually on the same task**

***Session 6:*** In the sixth session, students completed notesheets about salary and tax related to their chosen jobs. These notesheets required students to apply concepts they learned in the Mr. Needy salary activity. As in a previous session, a Jamboard was created for students to share helpful websites.

***Session 7:*** In this session, students worked on the final task of this assignment, which required them to write a report to share their learning. In this report, students explained why they chose their careers, what jobs they chose and why, as well as calculations of their gross and net annual salaries.

***Session 8:*** Students continued working on the report, Figure 7, in the last session of Career assignment. In this session, Stephanie reminded students to demonstrate and explain all the necessary calculations, and also explain the ‘mathematics’ behind them. Furthermore, students were asked to add APA citations of at least five websites they used in their report.



3) Why did you decide to choose that job?

I chose this because when I am older I want to be a beauty chemist. To my surprise it was very hard to find that specific job I wanted in Canada, therefore I had to be a little flexible with my final decision. I ended up picking to be a Cosmetic Colour Technologist, because it was the closest thing to the job I was looking for.

4) What is the **gross annual salary** of the job you chose, in the province it is in? Remember to refer to “starting salary”, if necessary. Feel free to speak about any kind of salary increase based on experience in that job. *Show/explain all necessary calculations.*

The symbol for multiplication is the *: Ex: 2*4 = 8

The symbol for subtraction is the -: example 4-3=1

My gross annual salary is $60,000. The job did not give me a starting salary, but I am guessing that, that would be my starting salary. The province that it is in is, Ontario. I found this out because the pay is, $28.85 an hour and I would work 40 hours a week. (28.85*40=$1,154 a week) then I multiplied that by how many weeks in a year (1,154*52=$60,000) and that is how i figured out my gross salary.

**Figure 7: Screenshot from the final report where Cymbi is explaining the working behind the salary calculations of her beauty chemist job**

## Bedroom design assignment

The final component of the Lifestyles Project was the Bedroom Design assignment. In collaboration with Penny (the arts teacher), the Bedroom Design assignment consisted of five classroom sessions. For this assignment, students had to design a scaled 3D model of the bedroom of their dream apartment while taking into consideration certain constraints, see Table 3.

***Session 1:*** Like the Hobby and Careers assignments, Stephanie and Penny started the first session of the Bedroom Design assignment with a brief overview of the assignment description and rubric. Next, Stephanie guided the students through the process of designing a top-view drawing of a 2D blueprint or floor plan. Students had to draw a scaled drawing of their proposed bedroom design on graph paper once they learned the basics of top-view drawing.

***Session 2:*** In this session, Penny taught the students how to glue and peel cardboard to make a 3D model of their bedroom using the previously designed blueprint. By the end of this session, students started designing their 3D bedrooms, see Figure 8.

**Table 3: Constraints of the bedroom design assignment based on mathematics and arts**

| Mathematics Constraints | Arts Constraints |
|---|---|
| Use at least three (or more) pieces of furniture in the shapes of regular solids or regular decomposable solids. | Create bedroom sculptures using low relief cardboard techniques (inspired by Louise Nevelson's work) |
| Calculate the total surface area of the bedroom by adding the surface area of each furniture piece as well as the surface area of the bedroom floor base. | Make patterned paper for decorating the floor, walls, and surfaces of furniture pieces, use multiple non-brush tools to create a painted wallpaper inspired by Dominique Petrin's artwork |
| | Ensure that elements and principles of art such as balance, variety, shape, pattern and composition are considered when designing and decorating the bedroom. |

**Figure 8: Secondary 2 students designing 3D, scaled models of their bedroom designs with Penny's assistance**

***Session 3:*** Students continued working on their bedroom designs in the third session. For the surface area calculations, Stephanie provided a notesheet for students to use.

***Session 4:*** In this session, students continued to work on the assignment. Stephanie and Penny responded to students' questions and provided feedback to students as they made progress on their designs.

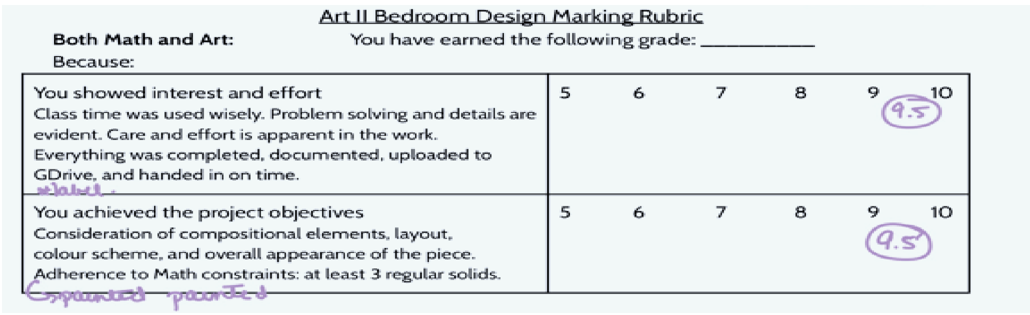
***Session 5:*** After making the final adjustments and modifications in their designs, students presented their work in this last session, see Figure 9. Students were asked to adjust their 2D top-view drawings based on their final 3D products.

**Figure 9: Examples of students' work from the Bedroom Design assignment**

## Student assessment in the lifestyles project

Each assignment in the Lifestyles Project was graded using rubrics, see Figure 10. The rubrics were designed by the two collaborating teachers of the integrated subjects. For example, the rubric for the Bedroom Design assignment (integration of mathematics and arts) was designed and graded by both Stephanie (the mathematics teacher) and Penny (the arts teacher). Generally, teachers introduced each assignment by discussing and explaining the rubric criteria. Whenever possible, at the end of each assignment, collaborating teachers met to review the rubrics and discuss students' performance.

Rubrics included both 'shared' and 'independent' criteria related to the two subjects that were combined (e.g., for the Hobbies assignment, mathematics, and science). Shared criteria included the general aspects of the assignment, such as the product, quality of work, and research. For each subject that was integrated within the respective assignment, the independent criteria included separate sections. As an example, the rubric for the Bedroom Design assignment included mathematics criteria such as surface area calculation and top view drawing, as well as arts criteria such as evidence of using low relief technique and elements.



Art II Bedroom Design Marking Rubric

**Both Math and Art:** You have earned the following grade: ___________

Because:

| | | | | | | |
|---|---|---|---|---|---|---|
| You showed interest and effort<br>Class time was used wisely. Problem solving and details are evident. Care and effort is apparent in the work.<br>Everything was completed, documented, uploaded to GDrive, and handed in on time. | 5 | 6 | 7 | 8 | 9 | 10<br>9.5 |
| You achieved the project objectives<br>Consideration of compositional elements, layout, colour scheme, and overall appearance of the piece.<br>Adherence to Math constraints: at least 3 regular solids. | 5 | 6 | 7 | 8 | 9<br>9.5 | 10 |

**Figure 10: Graded rubric where students' Bedroom Design assignment is assessed on shared criteria**

## Conclusion

In this descriptive paper, we presented the Lifestyles Project, an example IdME project from a Secondary 2 mathematics classroom at a Québec school. This project was designed using interdisciplinary approach by Drake and Burns (2004) and implemented using a PBL approach (Chi, 2021).

The disciplines of science, ELA, and the arts were combined with mathematics in an interdisciplinary, project-based format. Students first participated in the Hobby assignment where they chose a hobby of personal interest and identified and explored science and mathematics concepts associated with the hobby. In the Career assignment, students selected a future career, found a job in this career, and calculated the salary that they would receive (given required tax deductions). Finally, in the Bedroom assignment, students designed a 3D, scaled model of the bedroom of their dream apartment considering arts and mathematics constraints. Each assignment involved collaboration between the mathematics teacher and a colleague from another discipline (science, ELA, the arts). Rubrics were used for students' performance assessment, that aligned well with the interdisciplinary, project-based nature of these assignments.

We believe that the comprehensive example of the Lifestyles Project serves as valuable addition to the IdME literature and will help novice secondary teachers of Québec to adopt IdME in their classrooms. As the Lifestyles Project was particularly aligned with Québec's Secondary 2 mathematics curriculum, teachers in other contexts may need to make adjustments to make it suit their needs. Despite this, we still hope that the Lifestyles Project serves as an inspiration for those hoping to implement IdME in their classrooms. Learning from the examples in our paper, we encourage researchers and educators alike to further examine the ways to develop IdME projects and transform mathematics instruction to make it engaging, meaningful, and relevant for students.

## Acknowledgments

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# Preterm children's negotiation of mathematical identity in the figured worlds of home and the mathematics classroom

Rebecca Pearce[1]

*Identity research in mathematics education has increasingly been used to examine individuals' mathematics learning and experiences with mathematics in a holistic and agential way. My study answers the research question "how do children born extremely preterm negotiate their mathematical identities, and what insights may this identity negotiation provide about their mathematical learning and doing?" This paper will discuss exploratory findings from interviews with a secondary school student who was born extremely preterm, his parents and teacher, as part of my ongoing doctoral research. Within a case study methodology, I draw on a framework of figured worlds to examine student identity work and demonstrate that this identifying is complex and informed by the figured worlds of home, school, and prematurity.*

*Keywords: Mathematics, prematurity, identity, figured worlds.*

## Introduction and background

Preterm or preterm children, as a group, experience more difficulties with mathematics than term-born peers. For example, up to 50% of extremely preterm children, born at less than 28 weeks gestational age, show serious impairments in mathematics (Johnson et al., 2009) likely due to changes in brain circuitry impacted by lower gestational age (Klein et al., 2018). This is significant for many reasons. For example, there is a link between mathematics scores in early childhood and post-secondary school attendance and adult wealth in preterm children (Basten et al., 2015).

Current research studies into mathematical outcomes of preterm children employ the statistical analyses of standardized tests, checklists, and report scales, or the comparison of achievement levels of preterm children with term-born peers. While mathematics is important in both education and society, mathematics learning and achievement are complex and related to more than just cognitive ability. Current large-scale quantitative research obscures the individual mathematics experiences of preterm children and does not consider children's or parents' perspectives about learning and achievement. This research is decontextualized, in that it fails to consider children's family or school situations. Finally, the research is deficit-based in that it focuses on what preterm children can't do or how they are different from term-born peers, rather than examining their strengths or what they can do and be.

To examine mathematics learning and doing in preterm children in a different way, I have chosen to use identity as a lens to examine participants' experiences with mathematics. Research in mathematics education has increasingly looked at the concept of identity to understand the relationships that students have with mathematics (Darragh, 2013). This research "seeks to understand how individuals experience, perceive, and position themselves as learners and doers of

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mathematics" (Goldstein, 2018, p. 146) and how other people position and impose identities upon the individual. Examining ways in which students think about themselves in relation to mathematics can provide a more holistic measurement that goes beyond the analysis of students' mathematical reasoning and what they can and cannot do (Cobb et al., 2009) and focuses on the whole person rather than reducing learning to factors such as cognition or achievement (Darragh, 2013; Fellus, 2019). Since identity is related to persistence, proficiency, and learning in the field of mathematics (Bishop, 2012; Cobb & Hodge, 2010), it can help researchers theorize about mathematical learning and examine and understand people's relationships with mathematics (Darragh, 2016; Fellus, 2019). The concept of identity has been linked to learning through the adoption of socio-cultural theories, with the understanding that learning happens through social participation and the process of identity development (Esmonde, 2009). Mathematics identity as a construct can allow for self-authoring and agency, incorporates positioning and the impact of significant others, can provide insights about learning without using discourses of cognitive deficit, and can be used to examine participant experiences in relation to their social contexts (Fellus, 2019). In summary, the study of students' mathematics identities can allow us to access ways in which students see themselves and how they are viewed by others as members of a mathematics community (Anderson, 2007).

Mathematics identity research has been conducted with populations such as deaf and hard-of-hearing students (Goldstein, 2018), students with learning disabilities (Ben-Yehuda et al., 2005; Lambert, 2017), and those who are struggling in mathematics (Horn, 2008). Preterm children and the complicated ways in which their mathematics identities may be defined by existing, deficit-based research, by others, and by themselves make them a unique and unexplored group of learners. Preterm children represent a heterogeneous group whose differences may distinguish them from other mathematics learners. They have extensive and complex medical histories, which may lead to individual and family stress and trauma. They may also experience cognitive, attention, social, and emotional difficulties as a result of their preterm birth (Johnson et al., 2018) that may impact upon how they are identified by parents, educational professionals, and themselves as mathematics learners and doers.

## Research question and objectives

My overarching research question is *how do children born extremely preterm negotiate their mathematical identities, and what insights may this identity negotiation provide about their mathematical learning and doing?* There are four objectives to this research: To explore how secondary school students who were born extremely preterm negotiate their mathematics identities; to describe the contexts of home and the mathematics classroom in which these students engage in identity work; to analyze how positioning by others within the worlds of home and school influences/mediates mathematics identity work; and to examine how participants' mathematics identity negotiation and learning may be related.

## Theoretical framing

I have chosen to use the cultural model of figured worlds (Holland et al., 1998) as a framework to examine participants' identifying as mathematics learners. This framework takes up the view of learning as social, situated and involving a dialogic process of identity formation, what Holland and her colleagues refer to as "identity in practice" (p. 271). In this framing, identity is defined as how participants view themselves and their relationship to mathematics and how they are seen by others through their position and encounters in the social world. Identity is a process rather than a thing; it is something that is done or negotiated rather than what someone is.

Figured worlds, positionality, and space of authoring are different contexts of activity in which identities in practice are negotiated. Figured worlds are "socially and culturally constructed realm(s) of interpretation in which particular characteristics and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (Holland et al., 1998, p. 52). They are "frames of meaning" (Holland et al., 1992, p. 271) or frames of understanding. In the context of my study, figured worlds contain agents that shape social meanings and structures (e.g., parents, teachers, students), acts, which include mathematics teaching and learning, and forces, which are social categories based on conventions such as intelligence and motivation (Horn, 2008).

Positionality refers to the way in which people, in the case of my research, extremely preterm mathematics learners, fit within their figured worlds. Positions are shaped by social and cultural norms, expectations, and dynamics within figured worlds, so positionality carries dimensions of rank, power, and status (Holland et al., 1998). There are two ways that I conceptualize positionality in my research, how certain students position themselves as mathematics learners and doers and how they are positioned by others, so how students describe themselves within the contexts of the figured worlds in which they live and how they are described by different actors (e.g., parents, teachers, classmates) in those worlds. There is a link between figured worlds, positionality, and identity in that positions are shaped by figured worlds, and in turn this positionality contributes to identity work and helps shaped identity.

Individuals must negotiate their identities in a "space of authoring" (Holland et al., 1998, p. 63) which refers to the ways in which individuals take up or reject positions within their figured worlds. The space of authoring is a Bakhtinian concept in that it is "organized around the conflictual, continuing dialogic of an inner speech where active identities are ever-forming" (Holland et al., 1998, p. 169). Because individuals can adopt, reframe, or resist the different positions, values, and narratives that are offered to them within a figured world through this space of authoring, identity construction is a continuous, active and agential process (Sabbah & Heyd-Metzuyanim, 2021). Hull and Greeno (2006) liken the space of authoring to "voice" or "the ways in which individuals present and represent themselves to others and to themselves, thereby authoring and co-authoring their identities in the social worlds in which they participate" (p. 78). The key is that individuals can only answer to the positions and resources that are available to them, for example being smart (Hatt, 2012) or a low-level student (Luttrell & Parker, 2001). Therefore, an individual's identity in practice is developed through negotiation between available positions and how these positions are taken up or not.

Mathematics classrooms have been researched as figured worlds with specific learning environments and cultures (Darragh, 2013), practices, activities, and available ways of interacting (Boaler & Greeno, 2000) roles, identities, and socially constructed images of mathematics (Takeuchi & Liu, 2021), and curricula organized and represented in different ways (Horn, 2008). Classrooms are shaped and regulated by "visible and tangible things like course objectives, classroom rules, and grades, as well as factors that remain beneath the surface" (Caraballo, 2012, p. 53) and the way that the mathematics classroom is figured influences students' relationships with mathematics (Takeuchi & Ling, 2013). Individual families can also represent figured worlds (Cogan, 2016) with different agents, values and experiences with mathematics and prematurity. Families have different social and cultural expectations and definitions of success in school mathematics that can influence student behaviours, attitudes, and educational mobility (Zuckerman & Lo, 2021). In the context of my research, figured worlds contain agents who carry out tasks that shape social meanings and structures (e.g., parents, teachers, students). Figured worlds contain acts or actions, which include mathematics teaching and learning, and forces, which are social categories or expectations based on conventions such as intelligence and motivation.

Figured worlds has been used in different contexts as a framework to examine identity formation or identity work, for example with teachers (Avraamidou, 2019), Swedish construction engineering students (Gonsalves et al., 2019), high school mathematics students (Jones & Seilhamer, 2020), and science students (Wade-Jaimes & Schwartz, 2019), and as an aspect in a theory-driven literature review on identity in mathematics education (Fellus, 2019). Fellus (2019) used the ADAS model to classify research on identity in mathematics education in four different but interrelated dimensions including mathematics-related autobiographical identity, discoursal identity, authorial identity, and socioculturally available mathematics identities. While Holland et al. (1998) focus attention on contexts of learning through acts of apprenticeship, for example, Fellus (2019) turns attention to a way to understanding mathematical identity in a broader sense, which includes one's personal experiences with mathematics, opportunities one receives to develop an authorial voice in mathematics, how one is talked to and about as a learner of mathematics, and what available identities one can identify with vicariously.

## Methodology and methods

I used a multiple, comparative case study methodology (Stake, 1995) for my research. I chose this methodology to support a deep exploration of each individual participant, through detailed descriptions of case settings and situations and the use of multiple forms of data. It has also allowed me to examine similarities, differences, or patterns across cases. Participants were recruited through the pediatric follow-up clinic at a local hospital as well as through a posting on the Préma-Québec website. Inclusion criteria included children who were in secondary school during the 2021-2022 school year, who were born extremely preterm at a gestational age of less than 28 weeks, and whose parents had identified their children as having difficulties with mathematics in school. Four participants of different ages and educational backgrounds, five parents (four mothers and one father), and three teachers took part in this study. Ethics approval was obtained from McGill University, the hospital where participants were recruited, as well as the schools and/or school boards attended by student participants.

My main method of data collection is a series of semi-structured interviews, lasting between 30–90 minutes and conducted remotely over Zoom. Data collection started in the fall of 2021 and finished in July 2022. I have three interviews for all participants, parents, and teachers, except for one teacher who I interviewed only once, for a total of approximately 36 hours of interview time. I also have report cards or report card results from each student, IEPs for two students, and one psychoeducational assessment as artifacts. All identifying information from transcripts and documents, such as names of siblings and school names and locations, was anonymized. Pseudonyms were either chosen by or assigned to participants, based on their request.

I used a hybrid process of inductive and deductive data analysis (Fereday & Muir-Cochrane, 2006) to both interpret themes emerging from the data and to develop themes based on the model of figured worlds, participants' identity negotiation, and positioning by parents and teachers (Fellus, 2019). Interview transcripts were used to assemble background information for each participant (e.g., prematurity history, family background, participants' lives and interests). I looked for excerpts from participants, parents, and teachers that help explain and describe the figured world of home, prematurity, and the mathematics classroom that indicate how parents and teachers position students. I looked for interview excerpts or narratives where students author themselves as learners and doers of mathematics. I identified emergent themes that explore links between students' negotiation of identity or identity work, how they are positioned in different figured worlds, and how this might relate to their learning. These data are being compiled to develop case summaries for each participant and to develop themes that may be consistent across cases. This analysis is ongoing as part of my doctoral thesis work. The following section will present case information and analysis about one participant, William.

## Case study: William

## Background and preterm birth

William lives with his mother Christine and twin brother Scott. During our interviews he was in secondary 2 at a public English secondary school with about 1100 students, about 200 in secondary 2. He was 13 years old when our interviews started and turned 14 in 2022. William and Scott were born at 25 weeks gestational age. Christine had been on bed rest before she went into labour, and her children were delivered by emergency C-section. Christine described the chaos and stress surrounding the birth of her children, and after Scott and William were born, it was clear that Scott, who was delivered first, was experiencing complications. William spent two months in the NICU where he did not have any of the major complications associated with extremely preterm birth, such as intraventricular hemorrhage, lung problems, or feeding issues. Scott, however, spent three months in the hospital, and at nine months was diagnosed with Cerebral Palsy (CP). Today Scott has spastic CP and using Canadian Neonatal Follow-up Network (CNFUN) definitions, would be considered to have a severe neurodevelopmental impairment (NDI).

## Early education

William showed some early minor developmental delays, for example when he started preschool at two years old, he was not talking, and it took a long time for him to reach developmental milestones like being toilet trained. Christine said that this never stressed her out and that she expected William to be delayed developmentally because of his extreme prematurity. William attended a reverse-integration program in kindergarten so he could attend a year of school with Scott at his specialized school for students with disabilities. Christine attributed this to future difficulties that William experienced when he started attending a regular English elementary school starting in grade 1. William was never tested for a learning disability or neurodevelopmental disorder like ADHD in elementary school despite his teachers suggesting that he be assessed because he was struggling in several subjects, particularly those given in French. Christine was adamant that he didn't have a learning disability and shouldn't need to be tested, put into a category, or given a label to receive support in school. This decision was based on her experiences working with students with learning disabilities and her knowledge of the education system, and Christine saw William's elementary teachers as overreacting about what she considered to be regular developmental differences in her son.

## Secondary school

William was a student ambivalent about school, and said "it's not like I love school, but it's not like I don't like it, it's all right" when asked for his thoughts about going into secondary 3. Christine worked as an educational assistant at William's school. Both she, William's secondary 2 math teacher Josie, and William himself said that he had a small group of friends and was liked by his peers. In secondary 2, William's marks were generally in the high 60s and low 70s, with 80s in courses like gym, drama, and ethics. His lowest mark for both terms of the 2021–2022 school year was in mathematics, with 63% in term 1 and 65% in term 2. Throughout our interviews, William identified his favourite subjects in school as history and English, because they were easier for him to understand. He particularly liked English because of the teacher and how she made the material accessible and easy to understand for her students. William also said that he thought he was pretty good at writing. Christine echoed this, saying that William did well in English despite not liking to read and not being a huge reader. Despite her previous assertions that William did not have an attentional or learning disability, she suggested that William's dislike of reading "might be a little bit of an attention problem" but that she didn't investigate this potential difficulty with attention because she "always considered it fine" and felt it wasn't hindering him academically.

William's least-favourite classes were mathematics and science because he found them more challenging and the material more difficult to understand. Christine noted that William had started to experience difficulties in mathematics in grade 6, notably with word problems. That continued into secondary school, and in secondary 2 he struggled in mathematics more than she thought he would and required extra help. William's classes at his high school were relatively small, between 19 and 21 students, but he pointed out several times that his classmates' bad behaviour impacted on his learning, particularly in mathematics, because every class, students would need to be admonished or kicked out of class for their misbehaviour.

## The figured world of home

The main agent in the figured world of William's home was his mother. According to Christine, William's father was uninvolved in his education but despite this, William's father was still an important figure in his life. William told me that his father was good at mathematics because he used to use it for his job building fences, and because of this, "he (his father) just expects everybody to be good at it too." William said his father had higher expectations regarding mathematics than his mother, who "works with kids who are not good at mathematics" and is more "understanding where I'm coming from." William also told me that he thought that both of his parents felt that mathematics was important for him because he might need it in the future.

During our first interview Christine told me that she hated mathematics, had a horrible experience in mathematics in secondary school, and barely got a passing grade. She plainly said that she did not care about William's achievement in mathematics and did not think it was important for him to do well in mathematics if he passes in order to graduate from secondary school. Christine questioned why anyone would work hard at mathematics if they didn't have an interest in it unless it was necessary for a future career. Despite these negative discourses about mathematics, Christine told me that she tried to help William with his homework when she could and would try to help him study for mathematics tests. By the end of the school year, in response to Williams's new interests in architecture and health sciences, Christine's beliefs about the importance of mathematics shifted considerably.

Christine: Well, I go up and down. Like, it's, I think it's pretty important. Like yesterday, he was saying, he wants to be an architect. So I'm thinking you need to know. You need to be on that level, you know, so I don't know, maybe he'll continue on with it.

She also said that she hoped that William did not hate mathematics because it would be necessary for certain programs, and that she wanted him to know that those programs were achievable for him if he got extra help and "pushed" himself. Education was important to Christine, and she wanted William to continue in Cégep and university. She told me that William talked about going to university, so he envisioned himself pursuing higher education as well.

Christine's expectations for William and mathematics were focused more on his effort and how hard he worked than his marks in the class. She said she was happy as long as "he tried" and that as long as he didn't fail mathematics, she was fine with however he performed.

Christine: Yeah, like I just, because I tell him we're gonna be happy with high 60s, 70s. So you don't have to worry.

Christine had set clear expectations for William including going for extra help, not leaving the classroom unless he understood everything the teacher was explaining and completing all the review packages given to him by his teacher.

Christine positioned William as a student who needed to work hard in school and said that academics, especially mathematics, did not come easily for him. She characterized William as a student who could be distracted in class and didn't necessarily "catch on right away" because he wasn't always listening. There were some inconsistencies in this positioning. Although she said that William was "borderline" in mathematics, and that it was "quite difficult for him," Christine also said that "he's been doing well," referred to William as "average" in mathematics several times and told me that she felt that he was getting more confident and was starting to realize the importance of school, and this was reflected in his grades.

Despite being described by his mother as a student who was good at going for extra help when necessary, William was also positioned by Christine as a student whose difficulty in mathematics stemmed from his lack of effort. She also said that he procrastinated with mathematics homework, especially when he found it difficult, and needed to be pushed to do it. When I asked Christine about William's use of mathematics outside of school, she did not seem concerned about his everyday use of mathematics, for example during cooking or baking, and said that he was able to make connections between some of the topics he was learning in school and how they could be applied in more utilitarian settings.

The more we spoke, the clearer it became that Christine's positioning of William as a student and her views about his difficulties with mathematics were inexorably linked to his prematurity and to his brother's disabilities. Initially, when I specifically asked Christine whether she thought that William's difficulties in mathematics were related to his preterm birth, she said no and portrayed his struggles as similar to those of other students his age. Conversely, when I asked William whether he thought being born preterm had an impact on mathematics, he seemed unsure, and it was clear that this was perhaps the first time he had considered this question. First, he said “I don't really know anybody who’s, like preterm who does mathematics, but like, I don’t think it had anything to do with that.” However, as he thought further, he told me that maybe it might have an impact because extremely preterm babies’ brains are not fully developed when they are born, but that he didn’t think it would have a drastic impact on mathematics ability.

Christine said that because she worked with students with attention deficit and global developmental delay, in her opinion William didn’t have any signs of a learning disability and she thought that William “got out (of his preterm birth) without having any repercussions.” However, during our interviews Christine also said William’s elementary school teachers pushed her to get an individualized education plan (IEP) for William, and that she knew from her neonatal follow-ups that there was a higher likelihood that he could have attention problems or learning difficulties arising from his extremely preterm birth. Several times, Christine compared William to his brother Scott to explain why she was less concerned about him in school. For example, she said:

> Christine: Well, I think because I had Scott, high needs, I didn’t I didn’t worry too much about William. Although I probably think at some point he was, I worried maybe about school, you know, cuz I knew like attention deficit disorder, stuff like that.

Christine also said that unlike Scott, who has major disabilities, William “looks, normal, he is normal” and this might have made teachers forget that he could have some sort of processing difficulty. This was another example of Christine seeming to acknowledge the possibility that William’s difficulties in mathematics could be due to a subtle attention or processing difficulty or impairment, despite her previous assertions to the contrary.

## The figured world of school and the math classroom

The figured world of William’s Secondary 2 mathematics classroom was described somewhat differently by William, Josie (William’s mathematics teacher), and Christine. William’s description of a typical mathematics class depicted a traditional class structure correcting the previous days’ homework, worksheets, and class work from a worksheet package on whatever topic is being covered and completing this work in class or for homework. Most of this work was completed individually, though William said that occasionally students were allowed to work with another classmate and very occasionally in larger groups, but only when students “were on top of everything” and are ahead of the rest of the class. William recognized that he was more productive working by himself because he got distracted from the task at hand when working with others. When asked about evaluations in mathematics, William only mentioned tests, and said “well it’s not like we’ll go over the answers,” only when “the whole class did bad” or when many students had difficulty with a particular question. He said that tests were often given back at the end of the class when everyone was packing up, and just went into his class binder.

William often attributed more importance to the behaviour of his classmates in mathematics class, rather than the actual mathematical content covered in the class. He said a perfect mathematics class would involve changing the seating plan because too much class time was spent trying to get the students under control. He said that his class is sometimes behind because the students didn’t respect or listen to Josie, “fool around,” and took advantage of her, but also that she didn’t follow through on her word to discipline students when they misbehaved. He thought that Josie's expectations for her students related to behavioural norms in the classroom:

William: Well, I expect, I think she expects us to respect the teacher, and respect the classroom, as well as like, do our homework is a lot because my class don't do their homework, and so we always have to tackle that at the beginning of class. As well as like, study and try and participate so they understand.

Regarding the mathematics curriculum, William's perception was that many of the subjects covered in Secondary 2 were unnecessary for students to learn and his connection to mathematics as a subject seemed limited. He said that a perfect mathematics class would also involve "go(ing) over the curriculum that I would have to learn and change that up a bit and take out the stuff we don't need to learn." He said that Josie never talked about why certain subjects were important to learn, and that students in the class never questioned the relevance of the curriculum.

The academic achievement of the other students in the figured world of his mathematics classroom seemed to hold little significance for William. He was unable to identify any students who he considered to be good at mathematics. In general, he saw good mathematics students as having "really good" grades, studying hard, and modelling good behaviour in the classroom. He said he only compared his grades on mathematics evaluations with his friends, so he didn't know how other students were doing; his class seemed like a low-competition environment. Regarding his friends, William said "I think we all do the same" in mathematics with marks that are "up and down."

Christine had a generally positive view of the support offered to William. She said that teachers, including William's mathematics teacher, always made themselves available during lunch for extra help. William's school had recently started an after-school homework program for science and mathematics, and though William did not attend because neither he nor Christine thought it was necessary, the program was there if he needed it. Christine thought that the mathematics teachers at William's school were "pretty good" because they were always accessible to students through Google Classroom, and consistently posted extra resources online, like videos, for students to look at outside of class.

Christine also concurred with William that Josie's class management sometimes left something to be desired, however Christine placed far less importance on this than William. She said that Josie talked a lot and so students lost their focus and acted out, and that some students didn't like her as a mathematics teacher. However, Christine also said that Josie was completely organized, had her entire school year planned, knew the mathematics curriculum very well, and acted as a mentor to the other mathematics teachers.

Josie had been teaching at William's school for 12 years and had taught mathematics courses from secondary 1 to secondary 5. She described the secondary 2 mathematics curriculum as "really heavy" and a big jump from the content covered in secondary 1. She said that the main topic was algebra, presented in many ways through algebraic expressions, equations, patterns, and a heavy focus on geometry and the use of multiple formulas to determine the perimeter, surface area, and volume of shapes. These were some of the topics that William identified as being very difficult for him and as having little relevance to everyday life.

Josie's description of her teaching style and a typical 50-minute mathematics class was in alignment with William's characterization. Josie said she didn't use a textbook but rather put together her own resources, mixing and matching from different sources, and provided students with paper packages of work (also posted on Google Classroom), which she described as “a little bit old school.” Josie explained that a typical mathematics class would start with going over homework from the previous class and discussing questions that students found challenging. Then she would cover a new topic for about 20 minutes, and finish with practice examples and then homework. Most of the work is individual seat work except for the occasional review package that students were allowed to complete in small groups. Josie considered herself a strict teacher who wanted quiet when she was teaching and expected students to be quiet and listening to her when she talked, which contrasts with William’s perception of his mathematics class as a place where many students were not focused or listening. Josie described an evaluation system that was based on chapter quizzes, tests, and assignments, some of which students were allowed to work on at home over a period of several days or a week. She described a class environment in which she tried to support students as much as possible and gave them multiple opportunities to succeed, for example helping them with memory aids for tests and exams and answering questions over email or Zoom outside of school hours. All students in her class were allowed extra time during lunch to complete summative evaluations because she “wanted to know what they know” without the evaluation being a race.

Josie’s expectations for her mathematics students centered on practice, effort, and perseverance. Josie said she expected her students to “put their all into it,” and that students need to engage in mathematics and not just “coast,” and that “it’s not good enough for, just to show up in my class.” Josie felt strongly that work ethic and hard work are what distinguishes students in mathematics and that hard work has a major contribution to student success. When asked to describe her best mathematics student, Josie said:

Josie: I don’t look necessarily for the best mark-wise mathematics student, but again, I'm going towards work ethic and maybe that’s where I put my emphasis on. I don’t even know who I have as a best student.

Josie also spoke about the importance of struggle in mathematics, and how many students thought they aren’t good enough because they struggle “but struggling is part of the process. And that's what makes you good is that struggle.” Josie gave her students about 30 minutes of mathematics homework every night, each day, because she felt that students didn’t practice mathematics enough and that “if you don’t practice it, you’re not going to get it.” She constantly checked homework and called the parents of students who fail to regularly do their work. She said that “mathematics is practice and weaker students don’t, don’t get that practice in.” Similarly, Josie explained that she didn’t think all students can get 80s or 90s in mathematics and that some students have more of an aptitude for mathematics, but that “the difference between an enriched student who's good in mathematics and the regular students is their perseverance.”

Josie told contrasting stories about William’s hard work in mathematics class and his weak academic performance on evaluations, a distinction that is echoed by William in his identification as a mathematics student. William was positioned by Josie as a borderline student whose marks are very up and down, depending on the topic, but whose “hard work and his ethic and whatnot, is going to carry him through whatever he wants to do.” With respect to specific topics, algebra was difficult for William during the first term. Despite Josie giving multiple small tests on the same subject and not counting lower marks if students did better on subsequent tests, William’s marks in algebra were in the 50s. Josie said that his “reasoning and mathematics sense doesn’t seem to be there” and that he had difficulty with multi-step word problems, not always knowing how to approach the question and where to start mathematically. He also had difficulty knowing which formulas to use to find the surface area and volume of solids. Although Josie said she didn't think that William would be successful in the advanced Secondary 4 and 5 mathematics courses needed to pursue certain programs in Cégep, she also made it clear that she tried not to "write off" students because it is impossible to predict how students will end up using mathematics in their future careers.

Although his marks were borderline, Josie praised William's organization, independence, and effort. She said he always checked her Google Classroom page to stay up to date, regularly attended extra availabilities, and showed agency in asking for help when he needed it because "he wants to get it, he wants to be able to go home and do his homework." Josie said she never got the idea that William didn't like mathematics and that his hard work and attitude were noteworthy. Unlike Christine, who said that William could get easily distracted in class and sometimes had trouble paying attention, Josie described William as a model student who settled down to do his work, was independent in class, but wasn't shy to raise his hand when he had a question. She characterized him as being well-liked and having a good support base of friends that he could work with on mathematics during recess or at lunch. Finally, like Christine, Josie indicated that she didn't see William's challenges in mathematics as being very different from other students, saying "he's not strong in mathematics, but he's also not the weakest neither." She confided that she found the level of mathematics more difficult than when she or her own children attended secondary school, so mathematics difficulties were common among students.

## William's mathematics identity work

Throughout our interviews, I tried to ask William questions that would help me gain insight into his identity in practice (Holland et al., 1998) and how he performed an identity as a mathematics student. Three interrelated themes emerged around William's positionality as a mathematics student and his understanding of how this relates to both behaviour and academic performance, the tensions between his current and anticipated achievement in mathematics, and his ability to identify and connect with mathematics. There were both consistencies and contradictions concerning these themes.

Several different times during our interviews, I asked William what he thought it meant to be good at mathematics or to be a good mathematics student. Like the parallel narratives described by Josie, the answers that he gave included explanations about being a good student *in mathematics class* (behaviour, effort, and attitude) and being a good *mathematics student* (academics and grades).

Specifically, in defining what it means to be a good mathematics student, William said:

> William: To be good at mathematics would probably, in my classroom, probably be like that you're on top of your work, and like you're really like, organized and you get good grades like in mathematics and you know what you're doing and you're not just like, just like, try, just barely even doing good, like you're just actually trying to do good and stuff.

William also said that an adult who is good at mathematics probably did well in mathematics in school, went to college or university where they studied mathematics, and had a job that involves using mathematics. Although mathematics was William's lowest mark in secondary 2 and he found it difficult, he still narrated a relatively positive mathematics identity. He told me that he often enjoys doing mathematics and has fun with it and is perceptive enough to appreciate that the reason why he finds certain topics less fun is not because he hates mathematics but because it is frustrating to not understand.

William's identity is more aligned with performing as a good student *in mathematics class*. Although he admitted that he found some of the mathematics topics he was learning this year "overwhelming because all of the formulas we have to use…you have to know everything," he also told me "I think I'm okay at it (mathematics)" and:

> William: Um, well, I've probably, like I wouldn't say like, I'm like, the best but I wouldn't say like the worst, I'm probably like, in the middle between, like, good, and mediocre. Because like, some, some of the kids, they won't answer questions, or they won't share. And most of them like will like, fool around. And so I'm like, probably one of the better kids there. But like not, but more like behavior-wise than like, mathematics wise.

William made the distinction that he could still be a mediocre mathematics student academically but a good student behaviourally. William also said that he always did his homework in mathematics class and tried to participate by answering Josie's questions—some of the characteristics that he linked with being a good mathematics student—even though most of his classmate didn't do their work and weren't active class participants. This showed the importance that Williams placed on effort and attitude in mathematics class. When asked why it was important for him to do his homework, William's answer again indicated this distinction between "looking good" (behaviour) and "doing good" (mathematics grade):

> William: Well, like it used to be because I, like I, well, it still is but I just like I really, I want to actually like do well, I want to actually understand the topic. But sometimes it's like it's like that, but also, I want to like not like brand myself differently, but like impress them because like some of them like know my mother because she works at my school.

Conversations with William revealed tensions between William's desire and perceived ability to do better in mathematics, and his current level of effort and performance and changing understandings of how important mathematics might be for his future. William's narratives about his effort in mathematics were at odds with his definitions of a good math student as someone who tried hard. During all three of our interviews, William mentioned that he felt he could get better grades in mathematics if he put in more effort, that his current marks did not reflect his capabilities and that he had the potential to "do better." Despite previously defining himself as "between good and mediocre" in mathematics, he also framed his mathematics performance as subpar due to his lack of effort:

> William: Like, I think I'm, where I, like I feel like I could do better if I put in like, the time and the effort. But overall, like, I feel like I, like the grades are okay, I know I can do better. But if I actually, like tried, and if I really, really wanted to, but like, I'm in a good place now.

This was a consistent narrative for William, on multiple occasions he explained he could do better, that the effort that he put into mathematics was enough to get him a "good mark" or "to at least pass" but he could do better and "get that amazing mark" if he really tried, and that he had "the potential to do, actually do really good." When asked how his friends would describe him as a mathematics student, William told me that he thought his friends would say he is a good mathematics student, but not "at his fullest potential" and that he could "rise to the occasion" in mathematics if only he worked on it. Similarly, when I asked him what his parents said about his 63% in mathematics on his term 1 report card, William said his parents were not mad, but thought he could perform better in mathematics. In another conversation, William told me that his mother was the first person to know about his marks since she works at his school and with his mathematics teacher, and that she knows that he doesn't do "amazing," is mostly "mediocre" and that mathematics "isn't my strongest suit." Since William's descriptions of his efforts in mathematics closely mirror how he is positioned by Christine, it seems that he has assimilated her discourses into his authoring of his mathematics identity.

Despite saying he was in a "good place" with his mark, William also expressed some uneasiness about his 63% in mathematics in term 1 and expressed the desire to improve academically, telling me "it [his mark] isn't the best because I know I could do better. And I know I'm going to do better." When asked why he wanted to do better in mathematics William's response suggested an intrinsic desire to really understand what he is doing to make future mathematics classes easier:

> William: So and then I can understand it. And then, like, it'll just be easier if I ever have to, like, do it again, or relearn the lesson, or just have any like to use it like, any day, it'll just be easier for me to understand it. I just want to do it.

Despite this desire to get a better mark in mathematics and his statement that "I don't think like you're born with it [mathematics ability], I just think that it's the effort you put into it," William's continued insistence that he could do better if only he tried harder came across as somewhat defensive, a response to his confusion about why his participation, homework completion, going for extra help from his teacher, etc., were not translating into academic achievement. William alluded to this when he told me that he did worse than expected on mathematics evaluations based on the effort he put into doing his work:

> Willliam But my effort I'm putting in, like, when I try to do it, it just doesn't exceed to like my expectation of what I thought I was gonna get. And so I think I just over, like, I get overconfident or like confident, and then I just like, expect a higher expectation than what I really get.

William expressed some uncertainty about how he would or could bridge this gap between his current and future academic achievement in mathematics, and it seemed unclear what he could do to work any harder. During our final interview he told me that his goal for Secondary 3 mathematics was to succeed, get the best marks he possibly could, and be motivated to do well. When asked how he planned to put that plan into action and motivate himself more, William couldn't articulate how he was going to do this, telling me that he didn't know what to do right now, but that he would probably think of something to do later to improve. Similarly, when I asked about the mathematics, he would need to be an architect or a health science professional, he told me that by the time he got to those programs, he would "probably be able to do it."

Over the course of the school year, William frequently framed school mathematics as a subject that he often felt had little relevance to the real world and narrated a lack of engagement with school-based mathematics as a subject. He clearly recognized that mathematics has "real-world" importance and applications, but also told me that many of the mathematics subjects in secondary 2 wouldn't be useful outside of the walls of the mathematics classroom. This distinction between mathematics for use in the real world and mathematics that is irrelevant for real life is important, because William used this perceived irrelevance and lack of authenticity as a reason not to focus on certain topics:

William: There's no like one that I, like prefer the most, it's just I'd rather do like if I if I understand it well, I'd rather more focus on those, or like the ones that I'll actually have to use like in the real world, because I feel like if we just do the ones that we're not actually going to use in the real world, then I feel like we're just wasting time.

As an example, William said that finding the surface area and volumes of polygons and three-dimensional shapes were not topics that he thought were important for life, unless they were needed for a particular career, and that these topics should not be mandatory to learn in secondary school. Instead, the topics he deemed important are those that you'll "actually need" and will "use it every day," for example how to calculate costs at the grocery store or being able to measure and convert these measurements to different units.

During our first interview, William made it clear that he did not see himself as pursuing a career that involved the daily use of mathematics. However, during our last interview William expressed interests in careers in either architecture or something in the health sciences. He seemed to understand that both these jobs would involve the daily use of mathematics, for example when giving prescriptions or calculating costs, and did not seem worried about the mathematics that would be involved in those programs at the Cégep or university level. Considering that William identified someone who is good at mathematics as a person who studied mathematics after secondary school and has a career that involves the use of mathematics, this shows a possible shift in William towards a more positive affinity with mathematics as well as confidence for the future.

## Conclusion

This case study reveals that the mathematics identity work performed by one secondary student who was born extremely preterm is complex and informed by the figured worlds of home, school, and prematurity, positionality, space of authoring, and available identities.

The figured world of William's mathematics classroom was a place in which class conduct, effort, and hard work were valued over achievement and interest in the subject. William described himself as a competent mathematics learner in that he participated in class, completed his homework, and behaved well, and this is reflected in how he was positioned by his mother, Christine, and his mathematics teacher, Josie. Despite this, Williams's mathematics identities of competence were limited. During our first interviews William did not fully value school-based mathematics and was alienated by the topics in secondary 2 mathematics and the behaviour of his classmates. Nonetheless, at the end of the school year, he expressed a desire to improve his marks and potentially continue into post-secondary programs that required a more advanced knowledge of mathematics. However, William's past difficulties in mathematics and his teacher's partially dismissive view that his struggles were no worse than those of other students meant that certain socioculturally available mathematics identities (Fellus, 2019) related to academic achievement may not be readily available to him.

These limited identities resulted in tension and conflict in the way that William represented himself as a mathematics learner and doer and identified positively with mathematics—participating and completing all his work but confused about why this was not translating into academic success. To explain this discrepancy, William took up an achievement-motivation master narrative. Zavala and Hand (2019) refer to this as the ability of “individual students to be more successful in math and science by exerting effort and overcoming obstacles that arise in learning these subjects” (p 803) even though there may be factors beyond students’ control that impact on their learning and achievement in mathematics. Several times during our interviews, William alluded to the fact that he could do better in mathematics if only he worked harder and put in more effort. This narrative was also voiced by his mother, showing how William has taken up and integrated the discourses of significant others in his mathematics identity.

At home, William is a participant in a figured world of prematurity that is also informed by his mother’s view of mathematics as difficult and possibly less relevant than other subjects. In this figured world, Christine does not view William’s preterm birth as having a negative impact on his learning and achievement (“no repercussions”), despite her reference to subtle executive function problems that could be preventing William from reaching his full potential in math. One of the actors in the figured world of home is William’s brother Scott, whose significant NDI may have resulted in Christine minimizing William’s difficulties in school in general and with mathematics. This figured world of prematurity could be seen as a positive if parents view their extremely preterm children as no different from term-born children. However, it also means that if clinicians and psychologists are concerned about academic issues in extremely preterm children, there needs to be better knowledge translation (KT) or follow up with the parents and teachers of these children, especially for those who are at particular academic risk. My ongoing analysis will reveal whether this figured world of prematurity is consistent across cases.

In conclusion, initial findings from one case of a secondary student born extremely preterm highlight the complexity in his mathematics identity work, learning, and academic success. This complexity is not captured in the current research literature about preterm children and mathematics. Future analysis will provide more insight on this understudied topic.

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## Section 3

# INNOVATIVE APPROACHES TO MATHEMATICS LEARNING: MODELLING, EXPERIMENTS, DESIGN THINKING

# Incorporating experiments into the learning of mathematics

Amenda Chow[1]

*In teaching, experiments are a tool to complement classroom theory and enhance student learning. There is a lack of experiments in the teaching and learning of mathematics, especially at the undergraduate level. This article provides further insights, helpful suggestions, and examples on incorporating experiments into a university-level mathematics curriculum.*

*Keywords: Math experiments, active learning, mathematical modelling, exploration, experiential education.*

## What is an experimental math space?

## Background and motivation

In many university science, technology, engineering and mathematics (STEM) courses, there is often an experimental lab component that accompanies the lecture portion of the course. For example, students may go into their biology lab for a frog dissection, which complements a biology lecture about anatomy. Experimental labs are an important part of undergraduate student learning as they provide hands-on practical experience and offer a different perspective from the theories taught in lectures. Experimental labs also offer an opportunity for teamwork, exploration, design, creativity, and reflection. However, such labs are not normally used to support the learning of mathematics.

It has been noted for some time that there is an insufficient amount of experimentation in STEM classrooms, and this deficiency starts in early education (Atkin & Karplus, 1962). About twenty-five years later, Bishop (1988) notes his concerns about the style of mathematics teaching:

> From my perspective of a cultural view on mathematics education, I see four major areas of concern about the present state of mathematics teaching. They are the technique-oriented curriculum, impersonal learning, text teaching, and the assumptions which lie behind these. (Bishop, 1988, p. 7)

Hurst, Rennick, and Bedi (2019) note that engineering students are not receiving sufficient experimental design experience that will be required in their professional career and their suggested solution is to implement engineering design days. These are two days where classes are cancelled, and students work in teams to solve real-world problems that are built on the theory learned in their courses. Problems include constructing functioning furniture, building a dam with sustainable materials and creating software for a video game. Rather than engineering design days, in the mathematics discipline, an experimental math space is a possible solution.

Such a space is a physical laboratory, which holds equipment for students to conduct experiments related to the mathematical concepts they are learning. An experimental math space is meant to inspire exploration and experimentation in teaching and learning. Interested instructors can incorporate experiments into the mathematics they want to teach to their students. Whenever possible,

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experiments should relate to authentic real-world situations, and be relatable to students as this supports retention, engagement, grades, and recruitment (Campbell et al., 2008). Campbell et al. explain:

> Students who felt they learned more were also more interested in the course and felt that they, and others, participated more in the course. This same relationship occurred with the applications. For students, interest and learning are very tightly tied together and both are tied to participation. Other data collected about student learning and interest appear to support this. Real life applications can increase student interest and participation. (Campbell et al., 2008, p. 9)

Examples of mathematical experiments could be to use robots to model and simulate random walks in a math modelling course, collecting the measurements of a spring-mass system for an ordinary differential equations course, constructing physical graphs using manipulatives in a graph theory course, or folding origami shapes to investigate geometry concepts. In these examples, robots, spring-mass system, manipulatives, and origami paper would be objects housed in the experimental math space. More detailed examples appear later in this article.

## Implementation

The equipment contained in an experimental math space should be flexible, scalable, sustainable and inclusive. For instance, being mindful of purchasing equipment that is reusable and portable. As this is to be a welcoming and inclusive environment, there should be no cost for those interested in using the equipment in the experimental math space.

The actual experiments may be conducted by students in the experimental math space or may be taken outside of the space and into classrooms. Students may work in groups or independently when conducting experiments. Along with this, the experimental math space can facilitate other aspects of course development and completion. For example, instructors can use the equipment to explore concepts they want to teach and prepare their lessons. It can also be used by students who are completing course projects that benefit from an experimental component. Outside of the classroom, an experimental math space can also play a role in both student and faculty research projects, as well as for engagement activities such as math camps for high school students and STEM workshops for members of the community or as part of recruitment events during campus tours and department open houses.

In one type of mathematical experiment, students assist with the experimental design and conduct the experiment to collect and analyze the resulting data. For example, in one calculus course, students measured the time it took for a balloon to travel various heights and used the results to verify equations of motion (Gruzka, 1994). In another calculus course, to teach flow rates and ordinary differential equations, student volunteers were asked to drill a small hole inside a cylindrical container, fill it to a certain water level, and measure the time it took for the water level to go from 10cm to 3cm (Farmer & Gass, 1992). In both experiments, the materials (e.g., balloon, measuring tape, containers, etc.) are readily available household items, and inexpensive to purchase if needed.

There may be accompanying computer software needed to operate some of the equipment and collect experimental data. Using software is part of the experiment and part of student learning, and software also complements the mathematics being taught (Brown, 2014; Mamolo et al., 2011; Pudwell, 2017; Wohak & Frank, 2022). For Brown (2014) and Pudwell (2017), experiments are about exploring mathematics via computation using software. In particular, Brown explains,

> Experimentation was designed to focus on a specific set of goals: 1. Explore mathematical phenomena experimentally. 2. Detect patterns and provide mathematical explanations. 3. Explore mathematical thinking and process of conjecture. 4. Design and implement mathematical algorithms with computer algebra systems. (Brown, 2014, p. 283)

In Mamolo et al. (2011), undergraduate students were given clear plastic pyramids that could be filled with water. By filling the pyramid under various conditions, students explored spatial reasoning and geometric modelling in a tactile way. This was then followed by student work on the software 'The Geometer's Sketchpad' to investigate the rescaling of a two-dimensional pyramid (triangle), and to encourage connections between two-dimensions and three-dimensions.

Other types of mathematical experiments can be classroom demonstrations conducted by the instructor, or classroom activities in which students explore concepts using tactile objects. For instance, in a geometry course, Zome tools (Figure 1) can be used to build three-dimensional structures, which is helpful for students learning about visualization and abstraction.

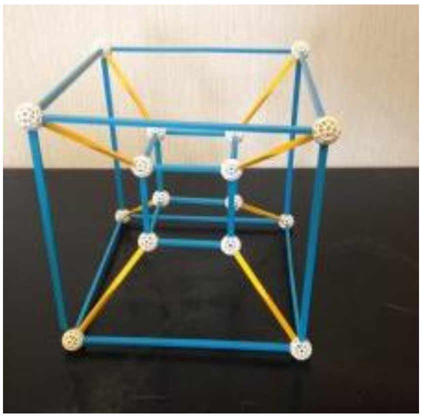

**Figure 1: A three-dimensional structure built from Zome tools**

Rather than Zome tools, students could construct geometric structures using found objects at home such as a malleable wire hangar. Encouraging students to recreate experiments on a simple scale using inexpensive objects and to do so outside the classroom is important because this conveys experimentation is a part of everyday learning (not just learning in a classroom) and inspires creativity, sustainability and flexibility. The focus of experimentation should be on the concepts taught, rather than on fancy expensive equipment.

Assessing student performance in an experiment is context dependent. For experiments that are in the form of classroom activities, like building models in a graph theory course, students may be evaluated on an oral explanation of their model. Classroom demonstrations conducted by the instructor may lead to class participation in online polls or surveys that record students' responses and grades. Experiments that require computer programming may be evaluated based on the students' code or graphs generated by the code. Experiments that have data collection may be assessed on the student's lab report, which addresses experimental design, math modelling, interpretation of the data, and results of the experiment. Brown noted the importance of this:

> Each lab report goes through the writing and editing process. Although time-consuming, writing and editing helps students understand the importance of careful reasoning and precision when explaining mathematics. (Brown, 2014, p. 288)

It follows that the student learning outcomes for incorporating experiments into the mathematics curriculum include:

(a) improving student persistent, retention and engagement,
(b) reducing math anxiety,
(c) supporting different learning perspectives,
(d) encouraging teamwork, communication skills and creativity,
(e) developing mathematical reasoning via experiential learning,
(f) practicing mathematical algorithms using computer software, and
(g) connecting mathematics to authentic real-world applications.

## Dissemination

Incorporating experiments into the study of university-level mathematics appears in various forms at different institutions. For example, the book by mathematics professors Banks and Tran (2009) was written based on several years of designing and teaching experiment-focused and modelling-focused math courses at North Carolina State University. They state the following:

> Our experience with this approach to teaching advanced mathematics with a strong laboratory experience has been, not surprisingly, overwhelmingly positive. It is one thing to hear lectures on natural modes and frequencies (eigenfunctions and eigenvalues) or even to compute them, but quite another to go to the laboratory, *excite* the structure, *see* the modes, and *take* data to verify your theoretical and computational models. (Banks & Tran, 2009, p. i)

The three experiments presented in Banks and Tran (2009) are: (i) heat transfer of copper and aluminum, (ii) vibrations in a beam, and (iii) acoustic wave propagation in a PVC pipe. In addition, there is considerable discussion on the mathematical modelling and computational simulations associated to each experiment. That is, their entire math course is designed with a primary focus on modelling, simulating, and experimenting.

In Canada, at the University of Waterloo, the Department of Applied Mathematics has several experimental math spaces each focusing on a particular field such as fluid dynamics, control theory, or mathematical medicine. These are primarily used for research and time spent in them is often done by graduate students.

The experimental math space in York University's Department of Mathematics and Statistics is focused on teaching and learning in undergraduate math courses (Chow, 2022). It is funded by a university teaching and learning grant. Its logo is shown in Figure 2, which conveys the interconnection between mathematics and experimentation.

**Figure 2: Logo for Experimental Math Space at York University**

## Examples of math experiments

All the examples of math experiments noted in this section are part of the collection housed inside York's experimental math space. They are used in several department math courses by various instructors. The discussions in each example highlight several learning outcomes presented in the previous section such as writing skills, use of software, working with others, connections between mathematical concepts and practical authentic applications, and reproducing experiments on a simpler scale.

**Pendulum Experiment.** The dynamics of a simple pendulum (Khalil, 2002, p. 5), illustrated in Figure 3, are described by the ordinary differential equation

$$\ddot{y}(t) = -\frac{k}{m}\dot{y}(t) - \frac{g}{l}\sin(y(t)) + u(t)$$

**Equation 1: Second order differential equation for the dynamics of a simple pendulum**

where $l$ is the length of the pendulum and $m$ is the mass at the end of the pendulum, and $y(t)$ is the angle of the pendulum from its rest position over time $t$. The parameters $k$ and $g$ are the coefficient of friction and gravity, respectively. The external applied force is $u(t)$.

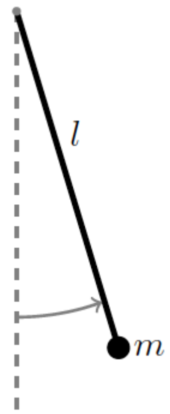


**Figure 3: The dynamics of a simple pendulum**

When a force is applied, the simple pendulum moves left to right in the plane. In three-dimensional space, the movement of the pendulum is more complicated. Shown in Figure 4 is a pendulum experiment designed by Canadian engineering education firm, Quanser. The vertical blue rod is the pendulum, and the pendulum is connected to a cube-shaped rotary servo base by a rotary arm. The cube base has a motor, which controls the movement of the rotary arm and pendulum. The black box, displayed in Figure 5, that is to the left of the pendulum, powers the movement of the pendulum and is the interface between the pendulum and the laptop on the right.

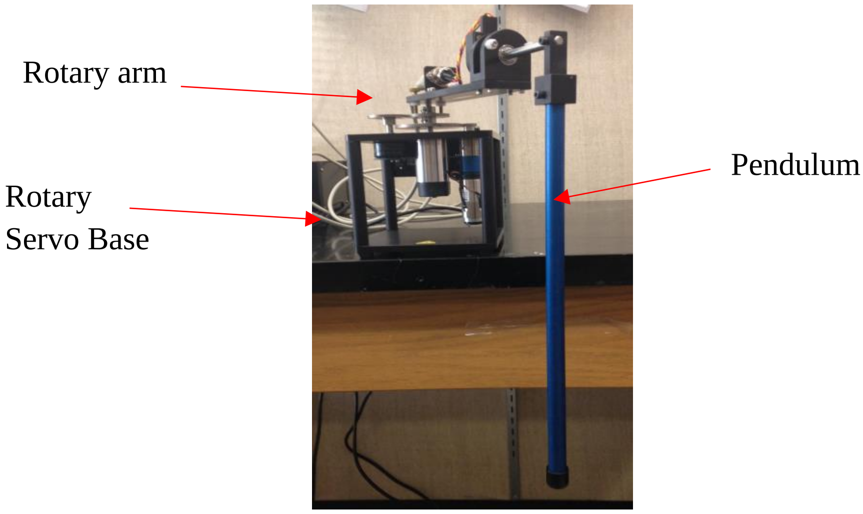


**Figure 4: Pendulum-rotary apparatus from Quanser**

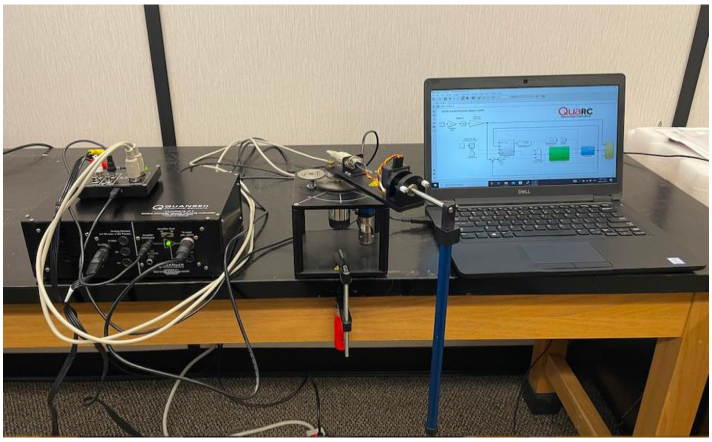

**Figure 5: Quanser's pendulum experiment**

The software used is MATLAB's Simulink, which uses block diagrams and connection lines to model and operate the various components of the pendulum's dynamics. A Simulink block diagram representing the dynamics of the pendulum is shown in Figure 6. Once the Simulink model runs, the pendulum swings, and this movement is recorded. A graph modelling the left right motion of the pendulum is shown in Figure 7.

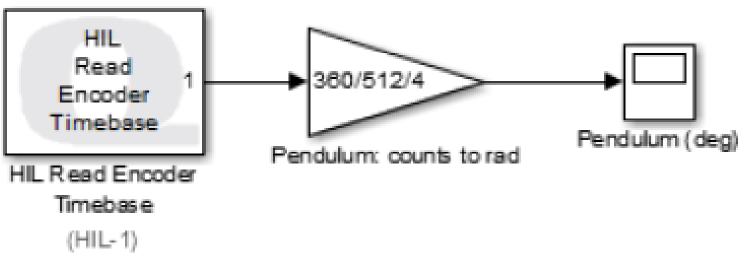


**Figure 6: A simple example of a Simulink model for the pendulum experiment**

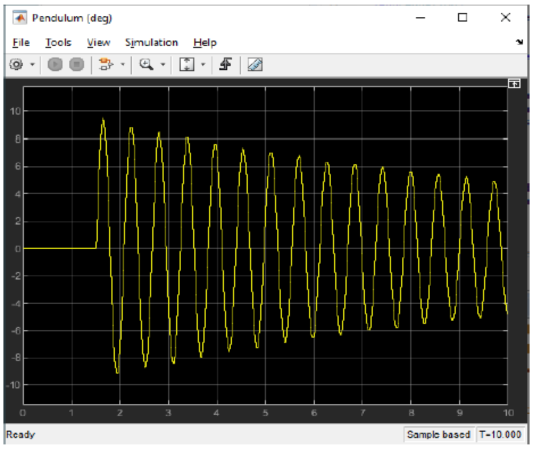


**Figure 7: The graph displays the position of the pendulum over time. Position is on the vertical axis and time (in seconds) is on the horizontal axis**

The pendulum experiment can be used for different objectives. For example, using damping to reduce oscillations, or applying feedback to invert the pendulum into an upright position. Prior to working with the pendulum apparatus, students form their lab groups (typically two to three students in each group) and are given a prelab, which prepares them for the actual experiment. The prelab gives an overview of the experiment, and a few preliminary questions to be answered, which are submitted by the lab group for evaluation, and returned with feedback to the students prior to the start of the actual experiment. Some examples of prelab questions for the pendulum experiment are, rewrite Equation 1 into a first order system of equations, linearize Equation 1, or derive a specific set of equations of motion. Prelab questions may also ask about aspects of Simulink or to list appearances of pendulums in everyday life. The mathematical concepts used in the experiment are taught in the course lectures prior to the prelab.

Each lab group works on the actual experiment for about two hours. There is normally a teaching assistant, lab technician, or instructor present to give guidance when needed. During this time, the lab groups set up the experiment, operate it in Simulink, and collect the necessary data that will answer questions posed in the experiment's lab report.

After completing the experiments, each group has one week to complete their lab report before submitting it for evaluation. Questions in the lab report typically require students to submit the graphs generated by their Simulink model, analyze the graphs, explore hypothetical scenarios, make connections between the experiment and reality, and address functions in the Simulink model.

Here is a sample question that appeared in the lab report for the pendulum experiment. The question is two-fold in that it asks students to compare the measured behaviour of the rotary arm angle in the physical experiment to the simulated response of the rotary arm angle produced by Simulink, and to make the same comparison for the pendulum angle. The expectation of the students' response is to generate corresponding graphs in Simulink (see Figure 8) and then to provide a written analysis of what is shown in Figure 8. From observations of the graphs in Figure 8, it is shown that the behaviour of the measured and simulated pendulum angles is similar but not identical, and there is a clear difference for the rotary arm angle. The paragraph below is a student's analysis**Figure 8**, which accurately describes the graphs, uses correct terminology taught in the lectures, and quantifies the difference in behaviour between the measured and simulated rotary arm response:

The model does not represent the system well. Inspecting the pendulum angle, $\propto$, we can see that the system closely matches the model, although it is slightly off. However, we can see that the rotary arm position, $\theta$, has some over and undershooting, but we can clearly see that as the system reaches steady state, it does not converge to what the model predicts. Although it does not converge, we can see that the phase of the model and the plant are the same, and the rotary arm angle is offset by approximately 1.2 radians from the model at steady state. Thus, the steady state error of the rotary arm is 1.2 radians.

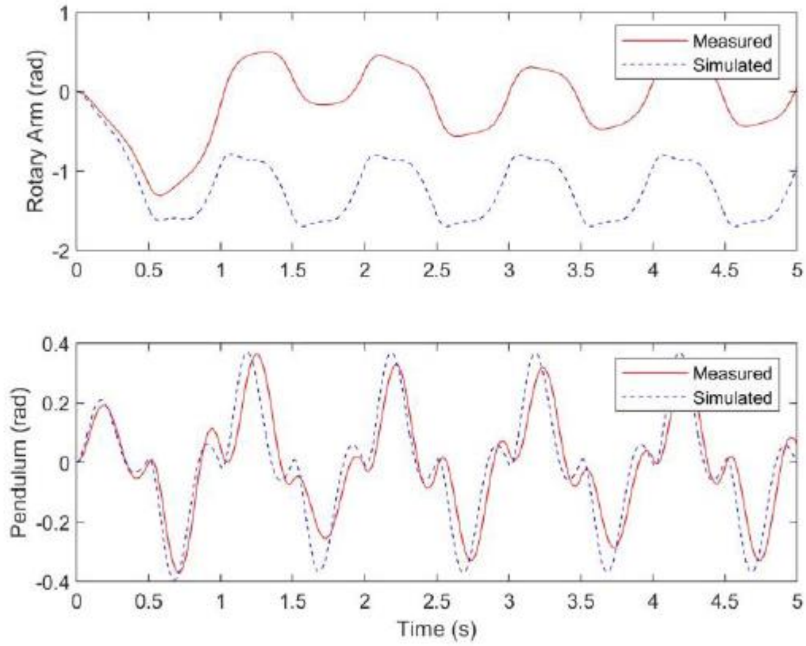


**Figure 8: Graphs show the behaviour over a 5-second time interval of the rotary arm angle and pendulum angle in the experiment (measured) and in Simulink (simulated)**

**Vibrations Experiments.** Vibrations are all around us. Sound is vibrations through the air. Earthquakes are vibrations on the surface of the earth. We feel the vibrations made by manufactured tools like jackhammers, cell phones and electric toothbrushes. Because of their periodic nature, vibrations are modelled by oscillatory functions; however, when sine and cosine functions are taught in a precalculus course, there is typically no mention of vibrations. In contrast, common topics taught in applied math courses, such as mass-spring systems, wave equation, resonance, damping, elasticity theory, Fourier analysis, and stability, are all related to vibrations. Consequently, there are numerous vibration experiments suitable for an experimental math space that can be incorporated into the teaching of these mathematical concepts.

A string-vibrator experiment (Figure 9) is ideal for visualizing vibrations. These vibrations are caused by the sine wave generator shown on the bottom left of Figure 9. The stable nodes (points of no vibration) in the standing wave are clearly visible. In this experiment, students are encouraged to explore string vibration by changing the frequency on the wave generator, or by varying the tension and length of the string.

**Figure 9: String vibrator apparatus**

The string-vibrator experiment inspires a simpler homemade version whereby one takes a bowl and wraps an elastic band around it, and then plucks the elastic band. The vibrations are clearly visible (Figure 10) even in this rudimentary form. A second useful and inexpensive demonstration of vibrations is with slinkies, which can be used to generate different types of waves (standing, travelling, transverse, longitudinal). There are also free online oscilloscopes and mobile phone apps like phyphox (Carroll & Lincoln, 2020; Staacks et al., 2018) that measure various properties of vibrations like frequency and amplitude. Phyphox can measure other quantities like position, speed, acceleration, and pressure, and this collected data can be exported for analysis onto a computer. An example of using phyphox and experimental data in a calculus course to compute the time of an object falling to the ground can be found in Chow, Harrington, Leung (2023). Phyphox was developed with educational physics labs in mind. Free online software and found objects at home are easily accessible tools for experimentation that enhances mathematical learning.

**Figure 10: Vibrations created by wrapping a rubber band around a bowl and plucking the rubber band**

The vibrating string (one-dimensional object) experiment in Figure 9 can be generalized to a vibrating plate (two-dimensional object). This vibrating plate experiment was created by scientist and musician Ernst Chladni (1756 to 1827). Ullmann (2007) states Chladni's experiment "is the first real effort to experimentally investigate the nature of sound" (p. 1).

A modern version of Chladni's plate experiment is illustrated in Figure 11. The square shaped metal plate is on top of a mechanical wave driver, which vibrates the plate. The driver is connected to a sine wave generator, which controls the frequency of the vibration. Sand is sprinkled over the square plate and as it vibrates, the sand collects along regions that are not vibrating (stable nodes). This creates aesthetically pleasing sand patterns called Chladni figures, which form interesting symmetries and patterns (Waller, 1961). The location of the stable nodes can be determined mathematically and simulated using software like MATLAB or Python. In an earthquake prone region, knowing the stable regions is beneficial as these are ideal locations for emergency shelters and hospitals. The vibrating plate experiment may be reconstructed from found objects at home such as a speaker, plastic wrap and a salt or pepper shaker.

**Figure 11: Vibrating plate experiment**

The vibrating plate experiment can be used as an example of mathematics in art (Fenyvesi, 2016; Fenyvesi & Lahdesmaki, 2017; Ornes, 2019). Different shaped plates and frequencies create artistic Chladni figures, which can be modelled by partial differential equations. For a circular plate, the Chladni figures have concentric circles as their pattern, and relationships between the frequency and number of circles have been developed, e.g., Chladni's Law (Rossing, 1982). The frequencies and sounds produced in the experiment can be an application for properties of sinusoidal functions and music. For example, Worland (2011) re-imagines this experiment with drumheads, rather than metal plates. Wohak and Frank (2022) present a lesson plan that showcases the mathematics and technology needed to compress audio signals. Chladni, whose love for music likely encouraged his lifelong scientific pursuits of sound, is considered by many as the founder of acoustics (Ullmann, 2007).

Quanser's flexible link apparatus is another example of a vibrations experiment. In Figure 12, there is a flexible link (object that looks like a metal ruler), which is free on one end, and fixed on the other end to a rotary servo base unit. The flexible link experiment is operated by Simulink and has the same physical set up as the pendulum experiment shown in Figure 5 except the flexible link (rather than the pendulum) is mounted on top of the rotary servo base. As with the pendulum experiment, lab groups working on the flexible link experiment submit a prelab, gather data, and complete a lab report.

The flexible link is similar in structure to objects like the wing of an airplane, a diving board, tree branches, the human arm and leg, etc. Vibrations can occur in each of these objects. In the flexible link experiment, once the link reaches its left most position (top image in Figure 12), it vibrates for a set length of time, and then it moves to its right most position (bottom image in Figure 12) and vibrates for another set amount of time. This repeats until stopped. Objectives of the flexible link experiment may be to investigate the behaviour of vibrations when the link is disturbed by an external force (akin to, for example, wind disturbance on an airplane wing), or to design a feedback model which reduces vibrations.

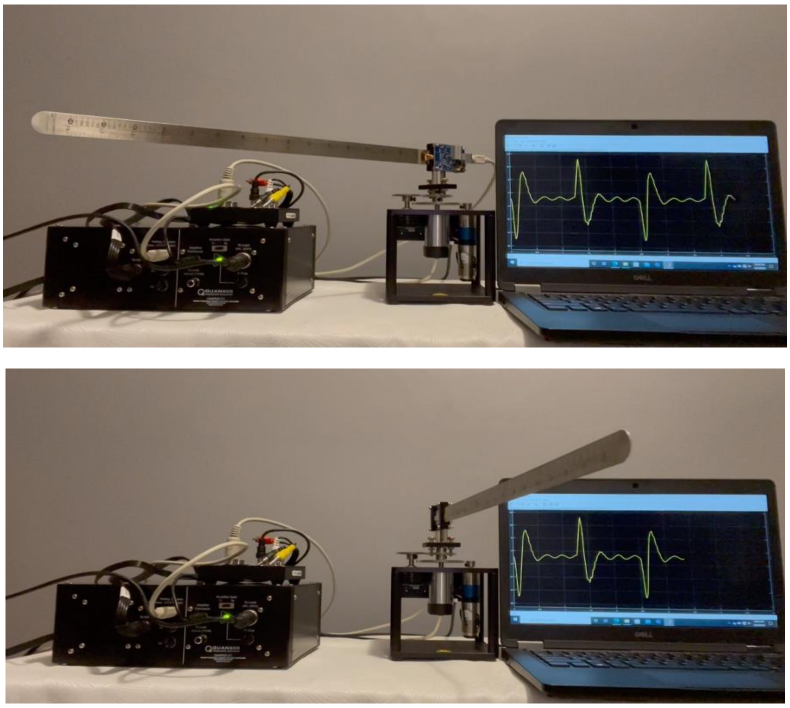

**Figure 12: Quanser's flexible link. Displayed on the laptop is the motion of the link as it moves from left to right. Its left most position is the top image and its right most position is the left image**

**Moment of Inertia Experiment.** In a multivariable calculus course, moment of inertia (also known as rotational inertia) is a commonly taught application for triple integrals (Stewart, 2016). For instance, the moment of inertia, $I$, of an object with volume $V$ and density $\rho$ is

$$I = \iiint_V \rho r^2 \, dV$$

**Equation 2: Moment of inertia formula**

where $r$ is the distance of the object to the axis of rotation.

An accompanying experiment is pictured in Figure 13 showing two objects of the same mass and same size, but that are differently shaped. The gray coloured object is ring shaped (hollow), and the yellow coloured object is disc shaped (solid). The objective of the experiment is to determine which object will roll down the plane fastest. The moment of inertia is affected by the spatial distribution of mass around an axis of rotation. In this case, the ring has most of its mass distributed around the edge, and this leads to greater rotational inertia (i.e., resists the change in motion more) than the disc. That is, the ring will move slower and hence, the disc will roll down the plane faster.

Addressing which object rolls down the plane fastest can also be answered by computing the angular acceleration of each object using Equation 2 and Newton's second law for rotation. The acceleration will be less for the ring, which supports the observation of the experiment.

**Figure 13: Two objects of the same mass and size (but differently shaped) race down the inclined plane. Which object wins the race?**

The experiment in Figure 13, while a simple classroom demonstration, allows students to explore many variations. For instance, what happens when it is two sphere shaped objects or if the height of the incline plane is changed or the two objects are of different size but with the same mass? This is also an experiment in which mobile apps like phyphox may be used to measure the timing, acceleration, and velocity of the moving objects.

## Reflections and looking ahead

In the course where the Quanser pendulum experiment was used, we note student feedback on the experimental aspects of the course. The course had eight students officially enrolled; however, one of these students submitted no work in the course and did not attend lectures or labs. Two of the students were graduate students, while the remaining were undergraduate students. This is a course cross-listed with engineering and all the students enrolled were studying engineering programs, except for the one absent student who was in a math program.

Towards the end of the semester, students were asked to complete an anonymous online evaluation of the course. Three students completed the evaluation, and there were two lab specific comments, which are noted below:

> I enjoyed the lectures and labs, I felt they were helpful in conveying the necessary knowledge on feedback control.
>
> I wish the labs could have involved more Simulink design, we changed a few blocks for the labs, but most of the model was already built for us.

Based on the second student comment, at least one student wanted their labs to be more challenging. In future offerings of the course, this is something that may be implemented as an optional bonus component for students who want to challenge their experimentation skills. We may also request students give an oral presentation of their experiments and findings so that they can practice more of their presentation and communication skills. However, in general, the addition of the labs in the course was beneficial to student learning.

Aside from the formal course evaluations, during the last lab of the semester, students who attended the lab (three in total) were asked for general feedback on the lab component of the course by the teaching assistant (TA). This was done in a conversational informal manner. The TA had assisted the students with their labs throughout the entire semester; that is, the students knew the TA. Here are the notes of the TA based on what the students were willing to share:

(i) enjoy group element,

(ii) good for conceptual understanding,

(iii) good balance between math and homework assignments and the lab component to apply their knowledge,

(iv) visual feedback and perception.

The last comment about visual feedback and perception seems to be missing some context, but it has been included for completeness. Overall, student responses are positive, but collected from a small group and anecdotal, so further rigorous study on the impact of experimentation on student learning in a math course is needed.

Through sharing the commonality of experimental labs, an experimental math space is an opportunity for mathematics to better connect with other STEM disciplines, and to possibly develop interdisciplinary courses whose curriculum is completely based on experimentation. The previous section noted engineering and physics examples, but other fields are possible. For example, a math modelling course is a natural place to teach connections between mathematics and biological systems, and to conduct corresponding experiments (Robic & Jungck, 2011). A geometry course may be an opportunity for experiments involving microscopes to learn about scale, position and measuring, and to explore the micro-world where artistic patterns like fractals and tessellations may appear. In another instance, a math and music course is an avenue to incorporate phyphox, vibration experiments, and computer software. In general, experimental math spaces can be a chance for artistic collaborations, which provides connections to such areas as group theory, geometry, nature, puzzles, computer graphics, architecture, textile design, etc. The possibilities are vast.

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# Rapturousness in makerspaces: Delight in construction

Olga Fellus[1] and Viktor Freiman[2]

*In efforts to bring change to their educational systems, New Brunswick, K-12 schools have been introducing new learning environments since 2014 (NB, Canada). These changes include the integration of makerspaces targeting STEAM disciplines and the formulation of new learning objectives such as the use of new technology and the development of 21st century skills (Freiman, 2020). This paper discusses part of a larger CompeTI.CA project (Compétences en TIC en Atlantique/ICT Competencies in the Atlantic Canada). We focus on the construct of simultaneous joy, delight, and enthusiasm to describe students' engagement in and commitment to building a shelter for their stuffed animal. We suggest the construct of rapturousness to describe makers' emotional expression. We present a case study featuring an engineering challenge that kindergarten students were trying to solve when designing a shelter for their stuffed animals at one elementary (K-5) school's STEM Lab. Aligned with the need to help children to navigate increasingly complex realities, and the need for students to have equal opportunity to study and appreciate processes of problem solving, innovative environments target learning objectives of a new 10-Year provincial Education Plan, which include improving numeracy skills for all learners as a key element in STEAM education, and enhancing learning in—and application of—the arts, science, trades, and technology for all learners (Province of NB, 2016). Reaching these goals would result from a better understanding of feelings of joy, delight, and enthusiasm when doing mathematics.*

Keywords: Makerspaces, delight, fun, joy, enthusiasm, rapturousness, engineering challenge

## Introduction

In early school years, the development of numeracy skill is recognized as critical to "providing a strong foundation to prevent gaps in student learning" (Province of NB, 2016). Numeracy skill necessitates sustained efforts throughout formal schooling and beyond focusing on problem solving, a strong foundational understanding of mathematics, and hands-on experiences. Against this backdrop, an increasing attention needs to be turned to community-school partnerships where educators are enabled "to provide learners with additional practical and 'real world' application of mathematical principles and concepts" (Province of NB, 2016, p. 11). This is particularly important in the case of makerspaces where strong partnerships with independent groups and associations, such as Brilliant Labs, allow schools to obtain resources and guidance from STEM-experts. Our study seeks to push to the fore the affordances of such collaborations, where mathematical principles and concepts surface in connection to other types of skills and capacity building, such as perseverance (Freiman et al., 2022). Our 2022 MACAS presentation focused on the concept of rapturousness that

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describes makers' joy, delight, and enthusiasm when engaged in constructing a shelter for their stuffed animal.

## Conceptual framework

We introduce the concept of rapturousness as an object of investigation in mathematics education in general and makerspaces in particular. We define rapturousness in a broad sense to include, at once, feelings of joy, accomplishment, and satisfaction. More on this in the next paragraph. Our work clusters the following concepts: rapturousness, engineering challenge, and design thinking in makerspaces and is couched within scholarship that highlights the inter- and transdisciplinary role of mathematics in STEM contexts (LeBlanc et al., 2022). This work also connects to a novel socio-ecological perspective of learning mathematics (Coles, 2023) that pushes to the fore mathematics education that is "marked by reciprocity, responsibility to others and by joy" (p. 19), that raises "questions of affect" (p. 22), and that taps "affective responses of engagement and even excitement) (p. 23). In this paper, our focus is on the theory and experience of the last words in these quotes—joy, affect, and excitement.

**Rapturousness**: While *rapturousness* is not very often used in the research literature, we see it connected to socio-emotional aspects of learning through movement and action (see De Freitas & Sinclair, 2014) in the making journey. We understand rapturousness as a feeling that combines triumph over an obstacle and self-management. This sense of rapturousness comes across in the writings of Pikionis (1989) who reminds us, "As we walk upon this earth, our hearts experience anew that rapturous joy we felt as children when we first discovered our ability to move in space—the alternating disruption and restoration of balance which is walking." (p. ). As we think about the concept of rapturousness, we see its connections with the Russian word v*ostorzheni*, which is used to describe—simultaneously—a sense of joy combined with delight. In English, the concept rapturousness semiotically offers a blend of elatedness, delight, pleasure, gratification, satisfaction, and triumph. The concept of rapturousness zooms in onto the intersection point of work in STEAM-based environments and affective aspect of the process of making through feelings of joy, gratification, and delight. This, we argue, corroborates the trend and direction to incorporate self-management through socio-emotional learning in education by allowing students opportunities and choices in what, how, when, and why they learn (Cristóvão et al., 2017). Against this backdrop, rapturousness, in our research, is operationalized through noticing expressions of sheer joy, high level of engagement, and exquisite delight. Specifically, we examine **rapturousness** in the context of mathematics education as intrinsically connected to students' work in other STEM disciplines.

We situate our work within the immense amount of growing literature on affect in STEAM-based environments in connection to mathematics education (e.g., LeBlanc et al., 2022) where affect and other denotations that are relevant to the concept, such as appreciation of aesthetics in mathematics (Sinclair, 2004, 2011) have been recognized as determining factors in students' learning and wellbeing. While emotions have been specifically identified as one of the three necessary types of engagement for developing mathematical thinking (physical and intellectual engagement being the other two) (Mason et al., 1982/2010), research that sheds light on positive emotions such as the concept of rapturousness that we are offering in this work remains dearth.

**Engineering challenges** provide students with opportunities to (1) design objects, (2) collaborate with others, (3) and learn science through processes of creativity while (4) using available materials such as cardboard. Designing objects helps students to see connections between science concepts and solutions to real-world problems (Sadler et al., 2000). Collaborating with others helps children develop understanding through interaction with and regulation of their environments (DeJarnette et al., 2021). Using materials such as cardboard allows for embodied activities, creativity, and imaginative play. It also contributes to students' wellbeing, enhances social and emotional development, while ensuring physical safety (Deed et al., 2022).

Understanding the affordances of engineering can also be rationalized through the lens of learning theories. The maker mindset is conceptualized through the theory of constructionism, which emphasizes child-led activities and interactions, where thinking processes are more valuable than end-products, and exploration and experimentation are positioned as core to the process of learning. Constructionist thinking draws on constructivism, another learning theory, that considers how learners construct knowledge through interaction, and experience and exchange ideas (Fletcher, 2007). Exploration and experimentation help students to articulate their mathematical reasoning, and make conjectures, and test to see if the conjectures are correct (LeBlanc et al., 2022). These stand in contrast to behaviorist conceptions of learning, which "dropped from [their] scientific vocabulary all subjective terms such as sensation, perception, image, desire, purpose, and even thinking and emotion as they were subjectively defined" (Watson, 1930, p. 5). As educational systems shift, educators today attempt to bring design-based learning experiences into the 21st century classrooms, which move to an explicit integration of social, emotional, and academic skills (Darling-Hammond & Cook, 2023; Zhang et al., 2022).

From this perspective, self-management and responsible decision making on the part of the students tie this work with literature on social-emotional learning (SEL), a new strand across contexts and curricula that is aimed at supporting the development of well-rounded citizens who can contribute "to the fulfilment of social demands" (Lee & Lee, 2021). One of the international trends we identify is to develop in students' skills of self-management and learning-related decision-making processes (Freiman et al., 2022). Research on makerspaces occasions such opportunities and supports positive SEL experiences (Darling, 2022). In the Canadian context, we notice an increased attention of provincial educational systems to SEL; for instance, Ontario introduced SEL in its new curriculum in mathematics (2020); New Brunswick (French) added socio-affective competence in its Exit Profile for Francophone Secondary School Graduates (MEDPENB, 2016). Our work on rapturousness suggests that makerspaces carry untapped potential in contributing to these concerted efforts of integrating SEL into school curricula.

## Data collection and analysis

Two groups of 20 kindergarten students and two teachers participated in the study. The researchers collected data during students' work using video recording and post-project interviews. The children were instructed to build a shelter to protect their stuffed animal from rain and high winds. In building a shelter for their animal, the children used different cardboard materials and child-friendly construction tools such as safe saws and plastic screw drivers and screws. The children worked in small groups of 2-3, with minimal teacher guidance. Twelve video segments were first analyzed observing the above-mentioned markers throughout the design of the shelters and the construction process. Then, initial codes were assigned to children's facial expressions and gestures first within each video segment and then across all video segments to finally investigate common aspects within and between all segments. Our initial analysis was guided by Hughes et al.'s (2019) design thinking process that includes iterative actions of asking, imagining, planning, creating, improving, and asking.


**Figure 1: Design thinking (Hughes et al., 2019)**

In a short-time activity like the one experienced by the children as they were working on their shelter task, we adjusted Hughes et al.'s (2019) design thinking cycle to reflect the design phases the children manifested in their collaborative work and created a three-phase process: Planning; Realization; and Testing and Adjusting.

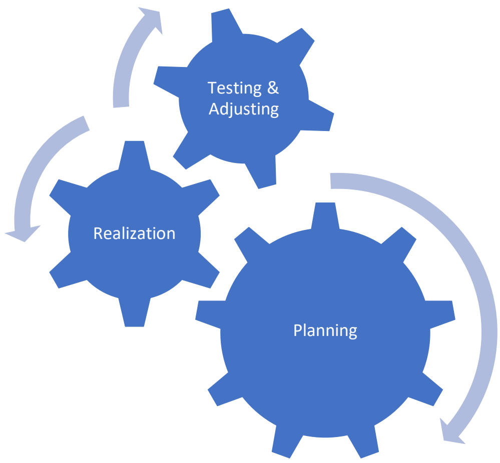


**Figure 2: Interconnected cogs in makerspaces: Three-phase process**

Through this process, the children first planned their project, carried out their plan, and tested the product to later adjust it if it did not satisfy the requirements of size, sturdiness, and protective properties. However, we noticed that this process was not necessarily linear. Instead, it was oscillatory as the children were moving among the different phases planning, building, and testing and adjusting, which is why we chose to represent this process in interconnected cogs rather than a cycle. Our analysis was inductive as we observed repeated markers of rapturousness during the time the children were immersed in planning, realizing, and testing and adjusting the design and construction of their shelter.

## A sample of the data and results

Analyzing the data as the children iteratively progressed through the inextricably linked cogs of planning, realization, testing, and adjusting in their makerspaces, we observed multiple combined expressions of rapturousness, i.e., joy, excitement, thrill, satisfaction, and triumph over challenges. These were captured through the children's gestures, intense focus on what their partners were suggesting, animated movement in the space they were using to build their shelters, and high levels of concentration that was evident throughout the project as they negotiated the design of the shape for their shelter, carried out the plan by cutting, adjusting, and manipulating objects, synchronized their actions and movements as they were working on building the structures, and testing and adjusting their structures through iterative cycles of trial and error.

Motivated by the task of protecting their animal from high winds and heavy rain, the children began planning their design by deciding on the shape and size of their shelter. For instance, Figure 3 shows two students discussing and estimating the height of their shelter in relation to the size and height of their stuffed animal. The experience of rapturousness captured in this phase was marked by highly animated gestures.

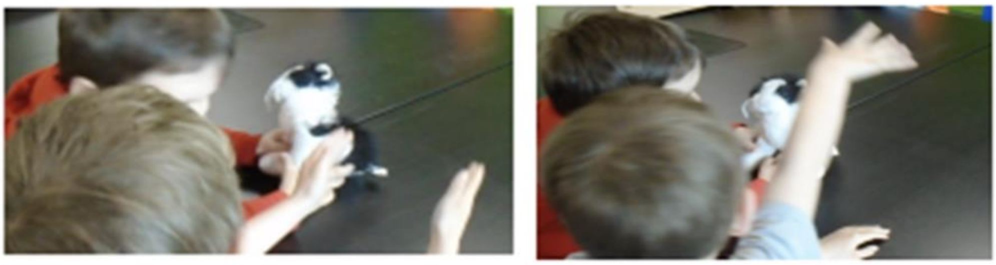

**Figure 3: Planning phase in the three-phase process in makerspaces**

In the *Realization* phase, children built a prototype of their shelter. As they were trying to construct the walls of their shelter, we noticed multiple expressions of rapturousness when children exuberantly tested their structures and exploded with cheers of excitement when it worked as planned. During the process of realization, the children exhibited synchronized movements, decision-making processes through logic and explanations as they were making conjectures and discussing how to fasten the joints of the faces of their construction to make their construction more solid, and effective collaboration with their peers throughout the work.

**Figure 4: Children show excitement and engagement as they negotiate the shape of the structure**

When the children decomposed the problem that they were trying to figure out. They negotiated the meaning of the constraints that they needed to take into consideration. They communicated their ideas looking attentively at each other, using animated body language, and exuberant gestures as they made conjectures and tested them out. These processes of working on a real-life problem through iterations of decomposition and negotiation of meaning, we suggest, are conducive to creating conditions for rapturousness. As the children negotiated the design of their shelter, they were immersed in efforts to design a structure that will genuinely protect their stuffed animal. Figure 5 showcases the children's intense concentration in their discussion.

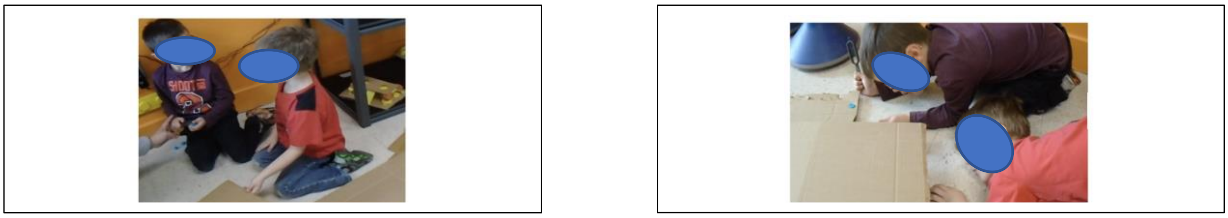

**Figure 5: Children immersed in the project as they negotiate the design of the shelter**

When the children checked if their respective constructions were solid, "waterproofed," and withstanding against "wind," they had to deal with several challenges. When they tested their construction, they were demonstrating efforts and capacity to adjust their structure by means of trial and error. Seeing their designed product becoming better, the children were visibly and audibly overjoyed as they could see their stuffed animal was protected and the shelter was solid. In post-project interviews, students explicitly referred to this sense of joy ("fun") coupled with perseverance as a factor contributing to their success.

Student: It was kind of hard to make the Lego challenge thing and the shelter building, but it was still fun.

**Figure 6: Children excited to see their designed structures function properly**

Students also valued their joint work through collaboration, which helped them to successfully carry out their plan. Without it, they admitted, this kind of an experience may not have been possible. The following interaction between the second author and the children conveys, we believe, this message.

Researcher: Last question, if you teach me, what are the qualities I should have to be successful as you were in the building activities? The STEAM activities?
Student 1: Maybe we should build with our partners and work together.
Student 2: Same as [friend's name]… that we should work together.

Expressions of joint rapturousness that we observed were collaborative, shared, and adaptive. Such feelings were manifested throughout the data. Indeed, at the conclusion of the activity, one team enthusiastically joint efforts with other peers to transport their shelter for a lion to their classroom.

**Figure 7: Children experience joint rapturousness while carrying their shelter to the exhibition**

## Conclusion

Our findings surface three contextually related and synergistically connected qualities in experiencing rapturousness through experiences that are experimental, collaborative, and creative. To wit, feelings of rapturousness are associated with students' experiences in minimally structured, non-prescriptive, and openly creative contexts. Experimenting through negotiations of tasks and decomposition of the necessary against available tools, young children experience rapturousness through discovery, trial and error, and moving ahead toward a shared goal. They get emotionally attached to the task at hand (i.e., protecting their animal) (Vongkulluksn et al., 2018). In this context of experimental work, we suggest, rapturousness is a catalyst in students' perseverance and staying on task (Freiman et al., 2022). It is not only the experimental context that can organically generate expressions of rapturousness but also the collaborative negotiation of meanings and actions, working together through acts of synchronization of movements, and jointly making decisions while working collaboratively to achieve their goal (Cook & Bush, 2018). Rapturousness and creativity is the third pair in this set of findings. It is through imagining and creating prototypes, bringing ideas together, suggesting new solutions, and reinventing the use of objects when children move from one shape to another, try to intuitively adjust objects they design (Li et al., 2019), that children solve a problem (Polya, 1945).

In particular, we attribute rapturousness to experiencing joy, satisfaction, and delight—simultaneously. To wit, as students were immersed in the iterative activity of planning, realizing, testing, and adjusting, we noticed their facial expressions, intense concentration, wide smiles, and animated gestures and movements. We distinguished between rapturousness at the individual level, which is self-driven, responsive, and directional and rapturousness at the joint level, which is collaborative, shared, and adaptive. These were noticed through students' verbal expression along with emphatic language, and expression of pride in what they engineered, constructed, and accomplished. Such feelings are an important part in the inter-, cross-, and transdisciplinary approach to mathematics (Robichaud & Freiman, 2022).

This sense of the association between experimentation, collaboration, and creativity is also supported by data collected in higher grades. When asked about next plans in making, after having completed a chainmail of 300 3D-printed pieces, one Grade 8 student responded:

Student 1: *I haven't thought of my next big project yet…they just kind of come to me when I lay in bed, like it will just come to me like, oh my gosh I need to do this! It'll be so cool! Like in bed trying to fall asleep, a project might come to mind and like this is going to be so cool I need to find a way to do that!*

In early school grades, young children learn through work on engineering challenges with imposed constraints and respond to problems as they emerge. Observing the children working on their engineering challenges, our study suggests that contexts such as makerspaces that spark students' creative minds, can engage them in an experience of rapturousness through **active exploration** of new technologies where they can produce new ideas and design and prototype objects that are **valuable for them**. Facing challenges, staying on task, and **working with others** are skills that are necessary to the development of creativity, perseverance, and more generally, a well-rounded, inspired, and **driven** person. Makerspaces can potentially offer fertile ground to investigate the development of mathematical thinking and reasoning while **supporting socio-emotional learning** through engagement and perseverance, which are interlaced, we suggest, with mathematical rapturousness. Our study has investigated such work in a small number of schools with a small number of students and teachers. Given that socio-emotional learning has become an important strand in teaching and learning in general and in teaching and learning of mathematics in particular, investigating the concept of rapturousness can potentially shed much needed light onto the experience of joy, satisfaction, elatedness, and delight in learning. Our finding merits deeper study on a larger scale for educational change in mathematics.

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# Teaching geometry with a Human-Centered Design approach

Gloriana González[1] and Saadeddine Shehab[1]

*Three high school teachers engaged in a virtual lesson study cycle to plan and teach a research lesson. The lesson required students to apply a Human-Centered Design (HCD) approach to mathematical problem-solving. Analysis of the pre- and post-lesson discussions answered the question: How do geometry teachers who are participating in a lesson study cycle use the Human-Centered Design framework to plan and teach a problem-based lesson? The findings show that the teachers made references to HCD in two ways: (1) as content in the research lesson's learning objectives for their students, and (2) as a practice for the teachers to develop the research lesson. The study has implications for engaging teachers in identifying authentic contexts for students to experience geometry problem-solving.*

*Keywords: Human-Centered Design, geometry, high-school math, lesson study, problem-based instruction.*

## Introduction

Calls for changing the high school math curriculum in the U.S. emphasize the use of authentic problems (NCTM, 2018). Geometry instruction can provide a special opportunity for students to apply math to real-world scenarios. Our study analyzes geometry teachers' engagement in creating and implementing a high school problem-based geometry lesson using the Human-Centered Design (HCD) framework (Lawrence et al., 2021). The framework emphasizes empathy and iteration for identifying and solving authentic problems. Three U.S. high school geometry teachers participated in a lesson study cycle following four steps as specified by (Lewis et al., 2006): (1) **studying** instructional materials, (2) **planning** a research lesson to observe student thinking, (3) **teaching** the lesson by one team member in a 9th grade geometry class, and (4) **reflecting** on evidence of student thinking during the lesson using videos from the lesson. Our research question is: *How do geometry teachers who are participating in a lesson study cycle use the HCD framework to plan and teach a problem-based lesson?* We describe how the teachers relied on the HCD framework to attain specific learning goals by designing and implementing a research lesson situated in the context of graphic arts. The context is intended to foster students' creativity in applying geometric concepts related to circles. Additionally, students had to apply math in design processes centered around stakeholders' needs and constraints, key aspects of the HCD framework. The teachers' lesson study engagement exemplifies their key role in embracing holistic and interdisciplinary approaches to education.

## Theoretical Framework

Human-Centered Design (HCD) is a problem-solving approach where people use design thinking processes and tools to identify the unmet need of a population to develop relevant and creative solutions collaboratively and iteratively (Brown, 2008). People using HCD rely on empathy and iteration (Brown,

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2008). When using the HCD approach, designers focus on understanding and collaborating with all stakeholders to identify problems, explore solutions, create prototypes, test the prototypes, and refine the solutions in iterative cycles (Brown, 2008; Brown & Katz, 2011; Dorst, 2011; Zhang & Dong, 2008). Designers implement practices such as interviewing people, identifying themes, communicating ideas, creating prototypes, and developing plans to bring final designs to the market (IDEO, 2015). Figure 1 illustrates an HCD framework that summarizes Human-Centered Design spaces and processes (Lawrence et al., 2021). The framework has five HCD spaces (Understand, Synthesize, Ideate, Prototype, Implement) and each space is composed of four processes.

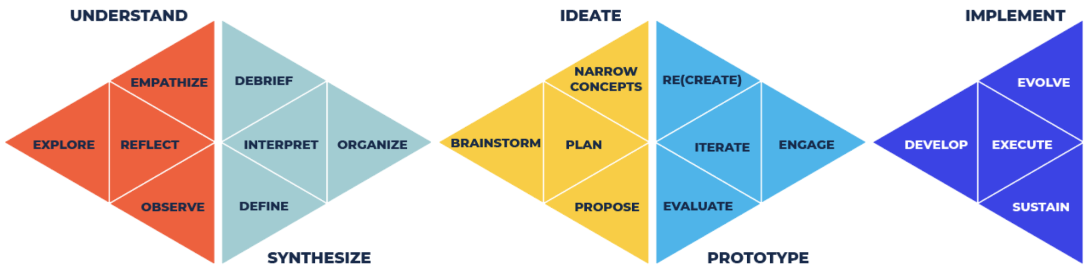


**Figure 1: The Human-Centered Design framework (Lawrence et al., 2021)**

Teachers are designers of learning experiences and instructional materials (Henriksen & Richardson, 2017). They can utilize the HCD processes shown in the framework to design and prototype students' learning experiences and instructional materials. Moreover, integrating HCD in problem-based mathematics lessons can positively influence students' learning. During these lessons, students can learn and apply mathematical concepts in authentic problem-solving contexts that feature multi-disciplinary collaborations. Students can also learn about and engage in HCD processes that can help them develop 21st century skills such as collaboration and creativity (Goldman et al., 2012; Koh et al., 2015). Prior research by Bush et al. (2018, 2020) in a $4^{th}$ grade math classroom shows that engaging students in design thinking results in their positive attitudes towards math. The students designed a prosthetic arm for a student to be able to use a keyboard. In doing so, the students created prototypes that applied math concepts and procedures while also developing empathy. Studies such as the ones by Bush and colleagues provide examples of how an HCD approach can leverage students' knowledge and experiences in math classrooms.

In this study, we explore how three geometry teachers used the HCD framework in the context of a lesson study. The lesson study team designed and implemented a problem-based geometry lesson that engaged students in applying properties of circles to generate a prototype of a restaurant logo that meets stakeholders needs and constraints.

## Methods

The study applies a design-based research methodology to design, implement, and evaluate four lesson study sessions (Easterday et al., 2018; McKenney & Reeves, 2012). Three high school geometry teachers were recruited to participate in a lesson study cycle during the spring 2022 semester. The teachers, all from public schools in a Midwestern state in the U.S., had never participated in lesson study before. All the teachers had prior experience teaching: Celia (13 years), Coral (2 years), and Dustin (24 years). We use pseudonyms for the study participants. The authors of this paper facilitated the sessions. To demonstrate HCD processes and foster creativity, the teachers studied prototypes of problem-based geometry lessons (see González & Deal, 2016). The prototypes were developed as part of another research study (González et al., 2023) and adapted by the authors to include HCD elements. The four sessions (1.5 hr/session) were conducted virtually using Zoom, Google Slides, and Google Forms. The online sessions allowed the teachers to participate after school by eliminating traveling time to a central location. The teachers could opt to have the camera on or off during the sessions and made use of the “chat” feature online. The availability of online platforms afforded the team to simultaneously edit the forms in real time when constructing the plan for the research lesson. Additionally, the online sessions allowed us to overcome the challenges of live observations since schools had visiting restrictions due to the pandemic. Nevertheless, the online sessions posed some limitations for data collection such as not having many opportunities for informal conversations, which are crucial for establishing rapport among team members. Additionally, the videos shown in the post-lesson discussions restricted the point of view to the way the first author positioned the camera when visiting the classroom. The research team selected the video clips for the post-lesson discussions. These decisions about where to position the camera and what videos would be discussed provided a different experience for the lesson study team than when conducting live observations. At the same time, post-lesson discussions with video can allow teachers to attend to student thinking (González & Skultety, 2018). During the first session, the team studied instructional materials and was introduced to the HCD framework. During the second and third sessions, the team planned the research lesson, deciding to focus on properties of circles (math content) and iteration (HCD process). The first author videotaped the research lesson in Dustin’s geometry classroom. During the fourth session, the team analyzed videos from the lesson focusing on identifying evidence of students’ use of properties of circles and iteration.

The sessions were video recorded. We created a timeline parsing the session into intervals, which is a unit of analysis noting changes in the activity structure (Herbst et al., 2011). In each interval, we identified (1) references to the HCD framework, (2) who made the reference (the teachers vs. the facilitators), and (3) the purpose or effect of the reference. For example, a reference to **“empathy”** by **a teacher** may have the effect of **changing the problem’s introduction** so that students feel empathy towards stakeholders **during the planning step**. Overall, the HCD framework specifies empathy and iteration as two key elements of a design challenge (Brown, 2008). We sought to understand how the teachers referred to these elements of the framework during the lesson study cycle. We coded independently a random sample of 20% of the total 75 intervals. However, we had difficulties achieving reliability in the first three rounds (60%, 53%, and 67%). After each round of coding, we met to resolve disagreements. After the third round, we developed a coding scheme for the references to the HCD framework (Table 1). We reached reliability in the fourth round of coding (86%) and the first author coded the remaining 20% of the intervals.

**Table 1: Codes for references to the Human-Centered Design framework in lesson study discussions**

| Code | Description |
|---|---|
| Yes | The facilitators or teachers make an explicit reference to the HCD framework or one of its processes or practices for the purpose of integrating HCD in the geometry problem or using HCD practices to engage in the lesson study cycle. |
| No | The interval **does not** include any explicit reference to the HCD framework or one of its processes or practices. |

## Findings

We introduce the findings in three parts. First, we provide an overview of the research lesson and the lesson study cycle to give the readers a sense of the context for the study. Second, we discuss the references to the HCD framework. Finally, we talk about the purpose of the references.

## Overview of the lesson study cycle and the research lesson

In the first session, the teachers examined three prototypes of geometry lessons using the HCD framework prepared by the research team. Table 2 includes a summary of the lessons, which provided the teachers with ideas that they used to create a research lesson in the second session. They decided to combine the math content of circles and geometry constructions in the “Designing an Analog Watch Prototype” and the context of “Designing a New Restaurant Logo.” The rationale for the selected math content was that students had prior knowledge of constructions. The teachers expected the students to apply properties of circles when creating their design. The rationale for the context about making a logo for a restaurant was the authenticity of the context, students’ love for food, and the opportunity for students to be creative. In terms of HCD, the teachers decided to focus on **iteration**. Specifically, they wanted the students to create and refine a prototype by incorporating peer feedback. The teachers critiqued the framing of the “Designing a New Restaurant Logo” lesson, which involved a fictional character, Dakota, who worked at an advertising agency. They stated that the students in the research lesson should not be living vicariously through Dakota and instead could be positioned as experts in the problem. The subsequent discussion led the teachers to frame the problem as one where the students were helping a small business owner of an existing or imaginary local restaurant. The teachers hypothesized that the students would be more invested with this new context and show empathy by relating to local needs in the community. Therefore, even though the teachers focused on iteration, they embedded empathy in the framing of the problem, an important characteristic of HCD.

**Table 2: Prototypes of geometry problem-based lessons with Human-Centered Design objectives**

| Title | Geometric Concepts | Human-Centered Design Goals Occasion | Problem |
|---|---|---|---|
| Designing an Analog Watch | Circles and geometry constructions | · awareness of HCD's problem-solving approach.<br>· reliance on connecting with people;<br>· understanding of users' needs as well as companies' constraints;<br>· use of math knowledge to make decisions on what qualifies as a feasible idea; prototyping a design. | Wristwatches are back! A new company is asking for proposals to do a unique production of wristwatches. They launched a competition, and the winner of the best design will earn $5 million dollars plus sale royalties.<br>As one of the participating design teams you decided to approach the task using the Human-Centered Design approach. You started your challenge by exploring the problems through talking to a specific population. You learned from interviews with millennials that many of them are interested in wearing analog watches (yes, the ones that are not digital). |
| Designing Backpack Patterns | Rigid transformations and triangle congruence | · awareness of the role of constraints;<br>· consideration of users' needs as well as executives' constraints;<br>· use of math knowledge to create designs that meet the company's constraints. | A backpacks company is trying to include its customers in the design process by giving them the option to provide their initials when they buy a backpack so the company can use it to create various patterns that can be tailored to the inside and outside of the backpack. To do so, the company launched a design contest, and the winner will get their design featured in their first advertisement. |
| Designing a New Restaurant Logo | Rigid transformations and congruence | · consideration of users' needs as well as executives' constraints;<br>· use of math knowledge to make decisions on what sketches meet the executives' constraints;<br>· use of sketches to collect and integrate feedback;<br>· appreciate sketching when prototyping. | Dakota works at an advertising agency. She and her team are currently working on designing a prototype of a logo for a new restaurant. They are putting together some sketches that meet the executives' constraints and they can use to communicate their ideas to the executives. |

## References to the HCD framework

After the introduction to the HCD framework in session 1, the teachers made references to the framework in at least half of the intervals. Table 3 shows the number of intervals with references to the HCD framework by the teachers or the facilitators. This finding demonstrates that the teachers appropriated the framework and used it to plan the lesson in sessions 2 and 3. Additionally, the teachers referred to the framework during session 4 when examining videos of students working on the problem. The facilitators made references to the framework in more than half of the intervals in all the sessions and diminished the references over time. It is possible that by making less references to the HCD framework, the facilitators allowed for teachers' agency in identifying connections to the framework by themselves.

**Table 3: References to the Human-Centered Design framework**

| Session No. | Lesson Study Step | Total intervals | Intervals with HCD references | Facilitators' HCD references | Teachers' HCD references |
|---|---|---|---|---|---|
| 1 | Study | 17 | 11 | 11 (100%) | 1 (9%) |
| 2 | Plan | 18 | 13 | 12 (92%) | 9 (69%) |
| 3 | Plan | 21 | 12 | 10 (83%) | 6 (50%) |
| 4 | Reflect | 19 | 11 | 7 (64%) | 7 (64%) |

## Purpose of the references to the HCD framework

We found two types of references to the HCD framework: HCD as content and HCD as a practice. On the one hand, the teachers' references to HCD when planning the research lesson aimed at students' understanding of geometry by applying the HCD framework. Therefore, the references to the HCD framework had the purpose of identifying ways to incorporate the framework into the research lesson. As stated earlier, the teachers decided to promote students' use of iteration in making prototypes of logos. They also embedded empathy in their framing of the problem for the students. In session 4, the teachers identified evidence of students' use of iteration when watching and discussing videos of the research lesson. On the other hand, the second type of reference pertains to teachers' engagement in HCD themselves by understanding, planning, and reflecting on the research lesson. The teachers used prototypes of lessons for planning. The research lesson itself could be viewed as a prototype for testing ideas about using HCD with students. In the following sections we show some evidence of these two themes from the data.

## HCD as content

We selected some examples of discussions across sessions to illustrate how the teachers referred to HCD as content. In session 2, when planning the research lesson, the team considered ways to incorporate iteration as students worked in groups of three (trios). Reference to "iteration," "empathy," and "prototyping" are noted in bold.

Celia: Okay, this could be a terrible idea. But what I'm thinking about right now is, as far as **iteration** goes, and if there would be enough time for this. So, let's say we give them, I don't know, 10 minutes to say, "Okay, you're in your trios, come up with the best logo that you can in 10 minutes." And then, there was in the description of the task, it says, "Finally, propose your sketch for at least two other groups. Collect their feedback and make at least one modification." I wonder if either in addition to it, and instead of that, say, "Okay, at the end of the 10 minutes, you're passing your logo off to the next group." And like, that's their starting point. And their goal then is to improve the logo that you started. And we do another 10 minutes, or maybe they don't need 10 minutes this time, because they're not starting from scratch. And now they get, I don't know, five minutes, six minutes, whatever the time is, and they have to improve it. And then it's like, "Okay, we're going to do **one more pass**." And, so now they get this third logo from a different group, and **they have to improve it** one more time. I don't know. I don't know if we have enough time for that. But...

Dustin: I like that idea. I'd be okay with **doing it once**. I don't know. Once they get the design that has been improved upon once, they might get bogged down in, "Well, what can I do to improve this? Because **it's already been improved, once.**" But to go back to Gloriana's question like, what's the number one thing we want to get out of this from the Human-Centered Design perspective? I could see it being just looking at the slides Saad has up, "**ways to empathize with stakeholders**," like you're trying to get feedback from multiple people, whether that's another trio in the room or I'm totally open to having them do a little bit of market research the night before, where they talk to family or friends about what makes a good logo. And then they bring that into the room that day. So, it doesn't take up time other than them dialoguing with the other two members of their trio. I think that would be a fascinating like, on-ramp to the problem. And they wouldn't know fully what we're doing. But they could probably surmise a little bit that they were going to be doing something with logos that day. I think it'd be cool for them to just ponder, "Okay, what makes a good logo an effective marketing ploy?"

Coral: So, I was thinking about, you know, when you said to pass it and have them make improvements, that to me fits in with **the ways to prototype a new design process**. Because in the real world, once you have a design, you often **pass it on to your superiors or peers** for them to make adjustments. And then kind of like, add on to what you already have. So, I feel like that would kind of connect the students to what happens in the job force, whether it's marketing, engineering, or anything of the sort.

In the discussion, the team debated the practical issue of how many iteration cycles to incorporate in the lesson. The team decided that students would ask another trio for feedback once, to avoid getting "bogged down." At the end, Coral stated that prototyping and incorporating feedback through iteration is an authentic practice in many job-related settings, thus adding authenticity to the task. In session 3, the team refined the activity so that students would perform one iteration in relation to other lesson activities.

Dustin: Are we going to have enough time guys to do it with two groups? I thought I raised, I'm just concerned like—I have in my mind, I have written on a piece of paper over here, like okay, we need this many minutes probably to do the design, this many minutes to pitch it to a group. I'm just trying to fit everything in. Can we get them to experience **the iteration part of it** just by having them run it by one group? Or I suppose now...no, never mind. Scratch that, that was my opinion. So.

Celia: I think just having one, one group, you know, that they have **look at it and give feedback.** I think that's fine. Especially trying to consider how much time there is. I mean, ideally, would they be able to get more than one group? I think if time wasn't an issue, like yeah, the more the better, right? But trying to do what's feasible in one class period, probably just one is good.

There were four objectives for the research lesson and two of them included prototyping and iterating: "Students become aware of **the role of prototyping** in Human-Centered Design" and "Students use sketches to **collect and integrate feedback**." The other two objectives made references to empathy: "Students understand that Human-Centered Design considers users' needs as well as executives' constraints." "Students use mathematical knowledge to make decisions on what sketches meet the executives' constraints."

During the lesson, the students used properties of circles to create a restaurant logo. Each trio made a prototype of the logo, received feedback from a peer who came to talk to the group, and then modified their logo. Figure 2 shows an example from a worksheet. The first prototype of the logo for a pizza restaurant had a comet inside of the circle. After receiving feedback, the second prototype included the comet as a tangent. Figure 3 shows their final prototype, which they presented to the class at the end of the lesson. The trio created a logo for a hypothetical restaurant, "Shooting Star Pizza." Their intention was to show 1 circle and its tangent, and use 2-5 colors, as established by the class in the initial discussion. The changes to the logo integrate the tangent to the circle as part of the design, minimize the crust, and refine the drawing of the tangent.

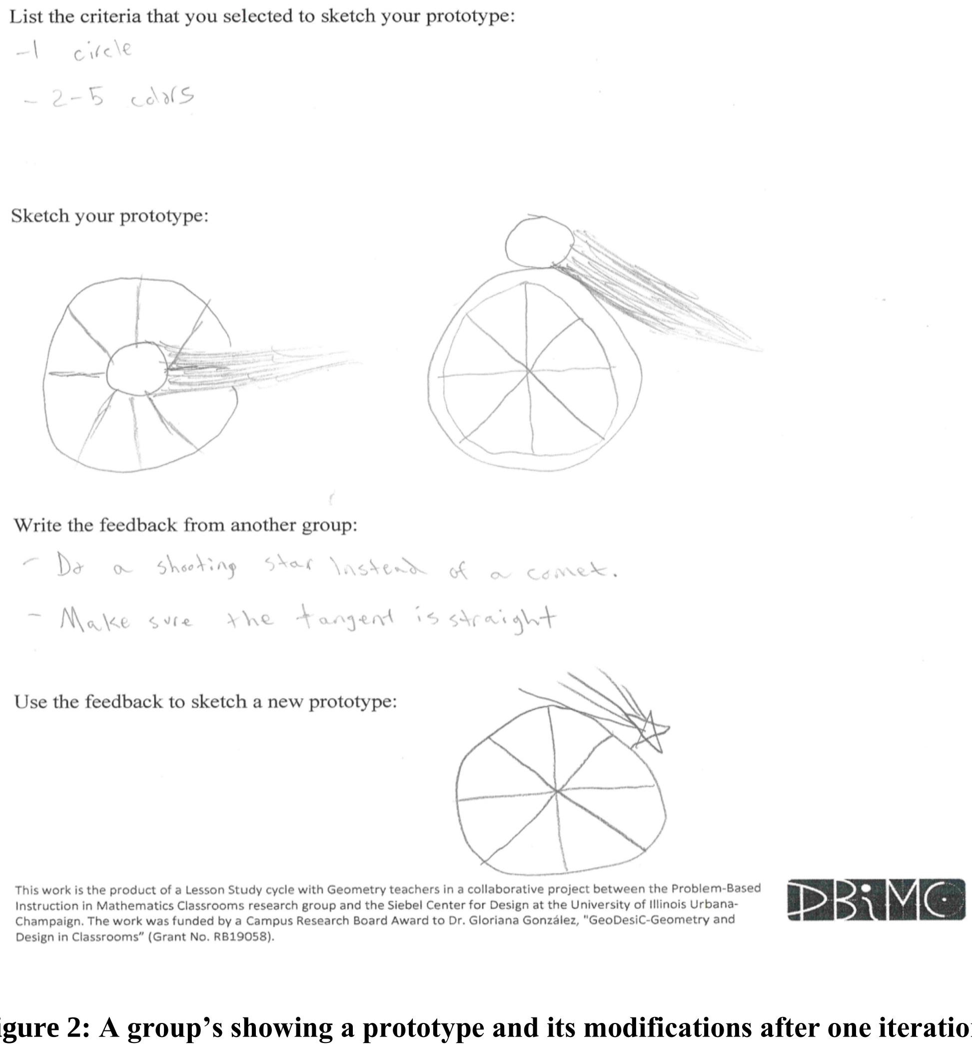


**Figure 2: A group's showing a prototype and its modifications after one iteration**

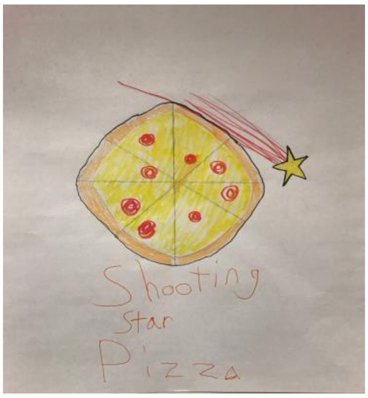


**Figure 3: Final prototype of a logo for "Shooting Star Pizza"**

In session 4, during the post-lesson discussion, the teachers identified students' engagement in the iterative process of creating a logo. They used the example of another trio that had been using a circle to create a logo for a taco restaurant.

Dustin: I will take the blame for rushing their development of the **second iteration** a little bit and tensions were flaring there at the end [laughs].

Gloriana: It worked, it worked.

Celia: I did appreciate though how the one student—it seemed like who like presented the **first iteration** to the other group and then was **bringing back the feedback**—she had said they, you know, they said to make the "B" more rounded. "It looks like teeth," they said. So, like **sharing that feedback**. And to me, like **taking the feedback** to then make the **second iteration** more visually appealing based on that feedback. I appreciated that.

The feedback was that the lettuce coming out of the taco and the letter "B," which was used to write the name of the restaurant in the logo, resembled teeth. The students changed the visual in response to the feedback. Celia noticed that the iterative process allowed the students to incorporate the feedback and improve their logo. Overall, the teachers used the HCD framework as content during the lesson study cycle. The examples show how the teachers integrated iteration in the research lesson's objectives and activities. Additionally, the teachers identified evidence of students' engagement with iteration during the research lesson.

## HCD as a practice

We selected some examples of discussions across sessions to illustrate how the teachers referred to HCD as a practice. In session 1, the teachers empathized with the students as they examined and discussed the three prototypes of the geometry problems. The bolded sections in the examples show that the teachers took the students' perspective as they reflected and commented on the prototypes.

Dustin: I was still in the process of reading the analog watch prompt. But I felt like at least two and a half of the…the situations were really strong from an HCD perspective, **really walking kids through the interaction and the interpersonal skills needed to make something happen in the real world.** And so, I saw that shining through loud and clear in all the models.

Coral: Yeah, same, like trying to read it **as a student, I think that the way that each of these tasks are introduced kind of gets them in the mindset** of, "Oh, yeah. Okay, this is why we're doing this." **Like they can see it as realistic, they can see it as having a purpose, not just, "Oh, it's another math problem," but actually the application.**

Celia: Yeah, I was going to say, a lot of the time kids ask me, like, "When would I use this in the real world?" And so, **each of these problems has, you know, an impact on…on some career, especially in design**, because a lot of times kids will often say, "Okay well, I can see how this would be useful like in engineering, or subjects in STEM, but how would I, how would it come into play if I don't want to do that?" So, I thought that was really neat.

After discussing the three prototypes of geometry problem-based lessons, the teachers demonstrated some synthesis and ideation practices to converge towards the restaurant logo problem targeting one math standard, circles as a geometry topic, and iteration as an HCD objective. The examples from session 2 show how Dustin started to narrow down the math standard for the lesson.

Dustin: **I would say if I had to pick one**, and I'm open to this everyone, it would be **the first standard listed in the watch problem. I think that's a nice blend of structure and open-endedness.**

Similarly, in the same session, Coral identified prototyping as the HCD objective since the problem already required students to empathize.

Coral: Yeah, I agree, I would say the ways to prototype in this design project, I think like, there is a sense of the empathy just from the problem statement itself, like you're asked to make something for someone else. **So, I think since that's already kind of integrated with the project, iteration would be a nice thing to focus on.**

## Conclusion

In our study, we found that geometry teachers can practice HCD to collaborate when designing problem-based geometry lessons. They can also integrate HCD elements and processes in geometry problems via lesson study. Our findings indicated that over the course of four online sessions, the lesson study team used HCD processes to design and implement a lesson with a geometry problem that encompassed both geometry and HCD learning objectives. It seems that the use of the HCD framework during lesson study empowered the teachers to collaborate and design an innovative geometry problem. The problem engaged geometry students in applying properties of circles and iteration to design a restaurant logo. The students had the agency of selecting circle properties to create their design. The constraints of the problem fostered their creativity by specifying elements to be included in the design challenge. The problem's context allowed students to develop some empathy and envision how to meet their stakeholders' needs with their design. The teachers appreciated seeing that the students selected a geometric property for their design and refined their integration of that property in their logo after building their prototype and receiving feedback from their peers. The iterative cycle within the HCD framework supported students' application of circle properties to an authentic setting of graphic design. In future studies, we would like to continue to explore how the HCD practices can empower teachers to design innovative lessons set in authentic contexts for students to enjoy and appreciate the beauty and relevance of math.

## Acknowledgment

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# Mathematics for multispecies' flourishing: A case for kolams

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*We have been developing the framework of Mathematics for multispecies' flourishing, which is an ethical and enactivist approach that recognizes the right of all species to flourish through the human study of mathematics and encourages respectful partnership between humans and other species (Tran et al., 2020). Kolam designs are geometrical shapes typically represented with dots, which are connected to each other through straight lines, loops, and curves. We interrogate some of the literature on kolams in mathematics education (Ascher, 2002; Chahine & Subramanian, 2017; Chenulu, 2007), which typically draws on ethnomathematical or culturally responsive perspectives (D'Ambrosio, 1990).*

*We revisit and resituate kolam drawing in mathematics education through a perspective of being for multispecies' flourishing (Khan, 2020) and using the language of variation theory (Lo, 2012; Marton, 2014), we argue that the external and internal horizon of the object of learning need to be continuously placed in relation to each other. This perspective on the kolam drawing practice serves as a pedagogical pivot (Ellsworth, 2014) that keeps the cultural and ecological significance always close at hand and in mind even as one deepens their mathematical exploration or appreciation. Our work responds to our wonder, "how do we create opportunities for passionate immersion and meaningful engagement with other cultures in ways that privilege a mindset of partnership and kinship over one of resource extraction and capitalization?"*

*Keywords: Mathematics education, multispecies' flourishing, ethnomathematics, variation theory, kolams.*

## Introduction

In this theoretical paper, we argue that Variation Theory (VT) and Ethnomathematics (EM) form a necessary and mutually enriching partnership that simultaneously advances multiple goals such as those aligned with equitable outcomes for diverse learners, cultural relevance, cultural respect, and humanizing mathematics education. We argue that *the external horizon of the object of learning is a critical aspect of the ongoing evolution of relevance and motivational frame for learning mathematics.* The lives of mathematical objects of learning that are embedded in cultural practices should maintain a harmonious relationship between both their ongoing cultural aspects and mathematical aspects. Variation theory also provides an important frame for the design of systematically sequenced and structured patterns of critical discernments that attends to the pedagogical dimension of ethnomathematics within school and other settings. We illustrate our ideas

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through the example of the practice of kolam drawing, which originates in Tamil Nadu, India, and has been previously explored in the EM literature.

*Steven:* *As I have explored the works of variation theory, I see the potential and importance to intertwine these concepts into the introduction of cultural learning within mathematics classrooms. I hope that in mathematics, the cultural significance gains traction, not only respectfully, but in a manner that supports and promotes students' learning. If variation theory can offer openings for students' ways of learning and knowing, there should be opportunities to connect their learning of mathematics activities beyond the classroom.*

*Olivia:* *My experience with kolam drawings started as a young bride whose husband worked in a small town called Karur in Tamilnadu, India. It was customary for the ladies of every Tamil household to get up early in the morning, sweep the front courtyard of the house with* ***coconut palm leaf stalk brooms*** *(recyclable) and sprinkle* ***water*** *to purify the swept area of the courtyard before drawing the traditional kolam designs. The purpose of sprinkling water is to get the rice* ***flour*** *stuck to the ground. People who have cows at home used to mix* ***cow dung*** *to the water and cement the front courtyard with the mixture before putting the kolams. In Kerala, we have the tradition of making beautiful patterns on the floor, aniyal (decoration) as they are known, the material used is wet rice ground and added with resin from* ***chopped okra*** *to thicken the paste.*

Kolams are drawn in front of Hindu households as a sign of **welcome**. I took lessons on kolam drawings from my friends, village women who lacked the opportunity to pursue higher education because of their gender and family traditions. Kolam drawings require mental concentration and prowess and are well connected with self-discipline (Personal experience; Ascher, 2002). I used to practice the kolam drawings many times on paper before putting them on the floor. I remember putting the dots in the sequence of 1-3-1, 1-3-5-3-1, 1-3-5-7-5-3-1 and this is done in precision without any ruler.

*Sreedevi:* *I have used sand drawings including kolams in work with elementary teachers for more than a decade (Khan, 2010). Teachers are always surprised at how much effort they take to make even simple images that are aesthetically pleasing and feel great satisfaction when they make increasingly intricate patterns. I am always careful to try to situate these traditions within their contexts and cultures. However, in math classes and the literature where cultural artifacts or practices form the basis for instruction, I have often found the cultural aspects to be rapidly dropped and never returned to as the mathematics takes precedence.*

Steven: *The work of teaching mathematics in schools is complex and challenging (Potari, 2012). To ambitious goals of high achievement, mathematical proficiency and equity for all learners, have been added goals related to humanizing, cultural responsiveness, and socio-emotional regulatory competence based on emerging consensus around how people learn. This consensus foregrounds the critical and necessary roles of contexts and cultural situatedness (National Academies of Science, Engineering & Medicine, 2018). At the same time, research on theories of learning* *reveal swarms of discourses (Davis & Francis, 2021) that intersect, interact, evolve, compete for attention, and influence approaches to, and beliefs about effective teaching.*

A critical convergent insight from neuroscience, economics, and education research is that attentional working memory (bandwidth scarcity) is a critical (non-volitional) limiting factor for learning (and learning to teach). This can be severely attenuated by factors such as physical, emotional, and economic stress due to poverty, racism, trauma, and marginalization (Mullainathan & Shafir, 2014; Verschelden & Pasquerella, 2017) but can be ameliorated by careful design considerations such as limiting unnecessary distractors, chunking into smaller pieces, and providing immediate corrective or evaluative feedback within a non-judgmental (growth-oriented) environment. This environment offers repeated and increasingly elaborative experiences, which elicits positive affective responses (see Figure 1). These findings are consistent with the design principles used in some types of digital games and puzzles (Khan & Rudakoff, 2019), some mathematics curriculum resource partners (Preciado-Babb et al., 2015) as well as Universal Design for Learning (UDL) frameworks (Lambert, 2021; Takacs et al., 2021).

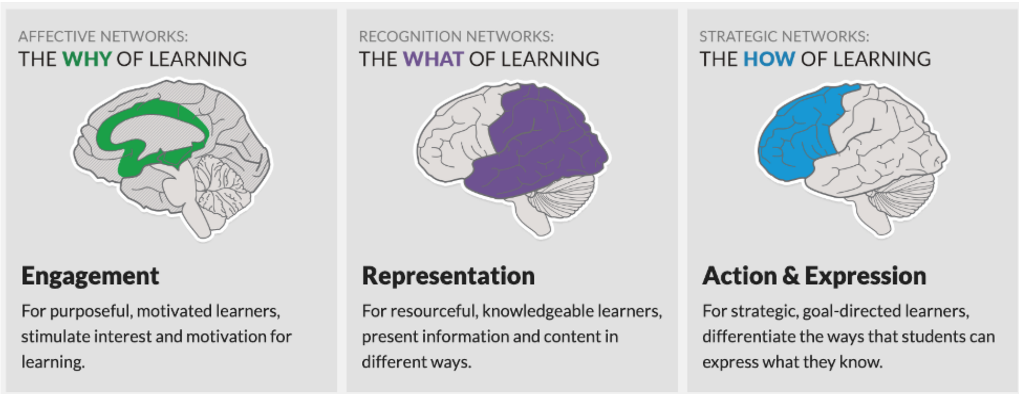


**Figure 1: Three networks influencing UDL principles—affective, recognition, strategic (Takacs et al., 2021, p. 33)**

In this paper, we bring together our understanding of the curricular and pedagogical design implications of variation theory (VT) and ethnomathematics (EM) in attempting to create a novel partnership between them that we believe will be useful to pre-service and in-service teachers who seek to include the cultural life-worlds of learners and their community into their classrooms in meaningful, impactful, and respectful ways while also advancing and remaining consistent with effective principles for learning mathematics. Our literature search and review found many articles over the last three decades that separately drew on ethnomathematics or culturally responsive pedagogical practices and those that drew on variation theory. We could find no work that *explicitly* partnered the two frameworks—though we acknowledge that in the description and design of some classroom ethnomathematical activities, we saw evidence of practices consistent with the broad principles of variation theory. Likewise, in some work drawing on variation theory we saw evidence of the wider cultural context of learners and communities being drawn in without explicit reference to ethnomathematics or culturally responsive pedagogy.

## Why partnering?

The idea of partnering draws on work of the Math Minds project (Davis et al., 2020) as well as work on multispecies' flourishing (Khan, 2020) both of which share an emphasis on an enactivist understanding of structural coupling and owe debts to indigenous ways of knowing, thinking, and being in relation to the other-than-human relatives—including mathematical concepts. The former involves work with teachers on partnering with a well-raveled learning resource that is consistent with their five principles of learning to improve teaching. They argue and present examples from JUMP Math lessons that systematically use structured and clearly sequenced patterns of variation of critical discernments that are sustained over multiple lessons and years. The work, however, presents as acultural and is focused on critical discernments of mathematical concepts and associated linking logics.

Our work starts from a position of cultural responsiveness (Matthews et al., 2022; Seda & Brown, 2021) and is consistent with recent findings from the National Academies of Sciences (1994) on How People Learn that contexts for learning matter, are essential for learning, more so for students from non-dominant cultures in education systems, and are not merely a backdrop or prop for learning. We take a more critical decolonial, anti-racist and eco-communal approach, which attempts to trouble capitalist-colonialist, essentialist understanding of 'resources' as something that are just 'there' for the taking and exploitation for individual or corporate profit. Indeed, the language of 'resource' in education is one that we find especially problematic in the context of the Truth and Reconciliation Commission of Canada (TRC) Calls to Action and moves to decolonize the University/Curricula.

Intentionally coupling EM with VT allows us to develop a strategy for working towards the ambitious goals of modern curricula and schooling in ways that feel honest, have a high probability of success, and, which honours the dignity and integrity of the learners we work with, the communities, heritages, and cultures to which they belong and contribute, and, who too must be invited into renewed vivifying partnerships with schools and education as they continue to transform each other. Thus, extending the work of Math Minds, we propose keeping cultural ideas central and find ethnomathematics a valuable partner along with variation theory in thinking about the design of learning experiences. Note, there are other potential theoretical and pedagogical partners with either member of this coupling, but they are not our focus in this paper.

As variation theory concentrates on the cognitive and design components and ethnomathematics focuses on the sociocultural components and human practices in multispecies worlds, we can couple aspects of working memory (from variation theory) with ethnomathematics and/or the cultural connections of ethnomathematics with variation theory. We will provide an example of kolam drawing to focus attention on how the use of variation theory and ethnomathematics might inform mathematics education and teacher practice.

## A brief introduction to ethnomathematics

Following Lubis et al. (2019), we take Ethnomathematics (EM) as expressing the reciprocal relationship between mathematics and/in culture (Lubis et al., 2019). It can be considered a "culturally specific practice performed by one cultural group seeking to make sense of another, often by reference to a specific conceptualization of mathematics" (Peralta, 2020). Katz (1994) pointed out that mathematical ideas have grown out of the needs of various cultures around the world, and it is important that students of Western nations are exposed to the mathematical practices in different cultures. The current Eurocentric challenges in mathematics education can be relieved through applications of ethnomathematics, as ethnomathematics creates opportunities to recognize the contribution of non-Western approaches to mathematics and to explore mathematics beyond the traditional framework of mathematical thinking (Hall, 2007). People all over the world have developed several mathematical methods consistent with their interests, religious beliefs, aesthetic, or recreational goals/purposes. Some cultures use arts and designs rich in symmetry, proportions, and transformations as part of their daily ritual, and kolam is one such example.

Ethnomathematics helps students understand the cross-curricular applications and identify mathematics in real-life situations. In the mathematics education context, ethnomathematics can introduce mathematical perspectives that focus on bringing diversity into the classroom through local knowledge and the idea that mathematics appears anywhere (Peralta, 2020). A multicultural approach to teaching mathematics can also guide students to comprehend the subject in an academic setting (in the classroom) and in an informal way outside traditional classrooms. According to Uy (2013), humanizing mathematics lessons helps to include all students and boost their confidence levels, promoting holistic learning of mathematics, and acknowledging the existence of "other" within mathematics. The practices of ethnomathematics have the ability to provide math educators with crucial resources that connect dominant and non-dominant forms of knowledge in ethnomathematics. The recognition of mathematics within cultural practices in conjunction with the discovery of different ways of thinking can be brought together as two perspectives of ethnomathematics (Peralta, 2020).

Ethnomathematics can serve as a bridge that connects the theoretical aspect of mathematics with lived experiences. Consider, for example, the increasing diversity of the population in Canada/North America, which has led to emphases on including learners' familial and cultural curricular assets in the formal curriculum. Ethnomathematics helps to overcome learning difficulties (Orey & Rosa, 2007). The authors elaborated on their assumption on the origin of modern mathematics as follows:

> Much of what we call modern mathematics came about as diverse cultural groups sought to resolve unique problems such as exploration, colonization, communications, and construction of railroads, census data, space travel, and other problem-solving techniques that arose from specific communities. (Orey & Rosa, 2007, p. 11)

Cultural variables have influenced students in their consideration of how they understand the world and interpret their experiences as well as that of others. In other words, culture influences the ways we gather and utilize our own mathematical knowledge (National Academy of Sciences,1994). Ethnomathematics helps students appreciate the contributions of their culture as well as that of others (D'Ambrosio, 1990; Joseph, 1991). As Freire (1986; 1998) mentioned, students are not containers to be filled with information, rather, teaching must involve the creation of knowledge and transference of information. Ethnomathematics is communal in the sense that all students in the class are significant and stay connected to their roots so that they develop resistance to harassment or domination and are equipped with the ability to engage in important concepts within mathematics, thus linking mathematics with its contexts.

However, incorporating ethnomathematics approaches has its limitations. One limitation is that ethnomathematics can privilege socio-cultural aspects over cognitive aspects of mathematics teaching and learning. Through the process of introducing the connection between mathematics learning and culture, often the progressive nature of learning itself is not regarded. In addition, the common method of practice and conceptualization for ethnomathematics is accomplished through only retaining the cultural aspects that are **deemed** relevant to the mathematical topics of interest. In this way, the mathematics portions are **'extracted'** from those cultural practices, often losing the meaning behind the tradition shortly after it is introduced. Therefore, the appropriate realistic applications of ethnomathematics practices should continuously incorporate various significances (traditional, ecological, familial, historical, theological, holistic) throughout lessons. Hence again, our semiotic signal in choice of the signifier *partner* rather than resource. A more comprehensive analysis of EM and some of the arguments within the field can be found in Khan (2008).

## A brief introduction to variation theory

Variation theory (VT) is a theoretical framework of learning and experience described by Ference Marton and several others (Marton et al., 2004). The variation theory of learning (Marton, 2014) focuses on the need for learners to notice/discern critical aspects of the object of learning. There is a core concept of *the object of learning*, which is situated in a context concentrating on "what" is learned and what students are expected to learn. In the framework of VT, the learners are drawn to contrasting by observing how something changes or is different, allowing them to mar Marton and Booth (1997) defined learning as the advancement of experiencing something in a new/different way, particularly at a moment(s) where there are differences in the structure of awareness. For Runesson (2005), "learning is defined as a change in the way something is seen, experienced or understood" (p. 70). In order to draw awareness to changes, sequences of patterns of variation and invariance are required and are called critical discernments. While developing these critical discernments, there are four necessary conditions of learning in VT to consider: contrast, separation, fusion, and generalization. We have summarised our understanding of the elements of Variation Theory in a series of concept maps presented below, which are also available online. [Note there are also linked concept maps that extend and deepen the understanding of the Object of Learning (OOL.),] the internal and external horizons of the OOL, and the core concept of critical features (or discernments in the Math Minds rendering). In the interest of space, these have not been included.

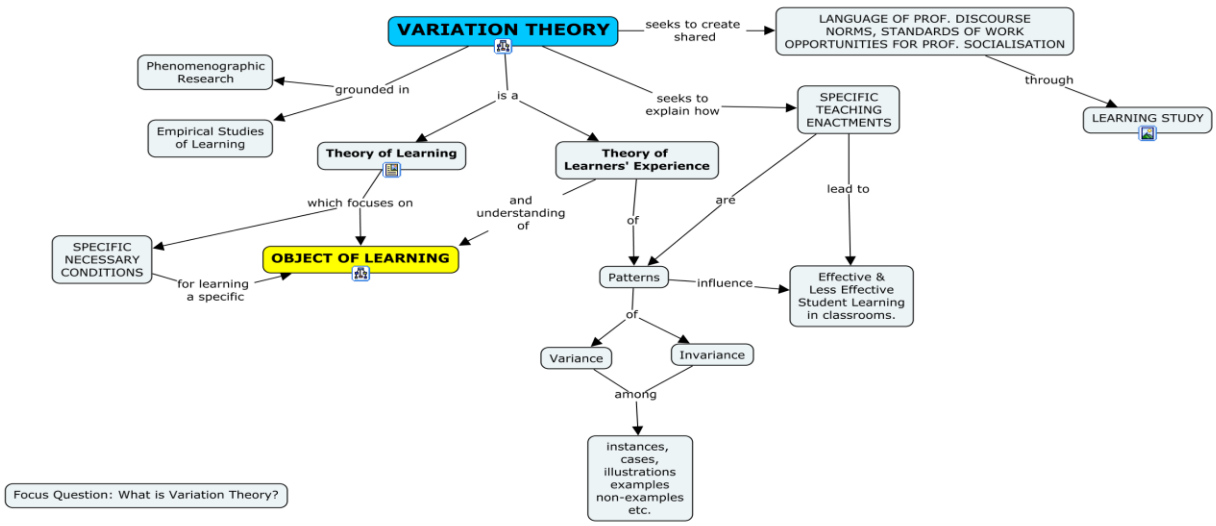


**Figure 2**: **Concept map on theories of learning and experience to attain the object of learning (Khan, n.d.)**

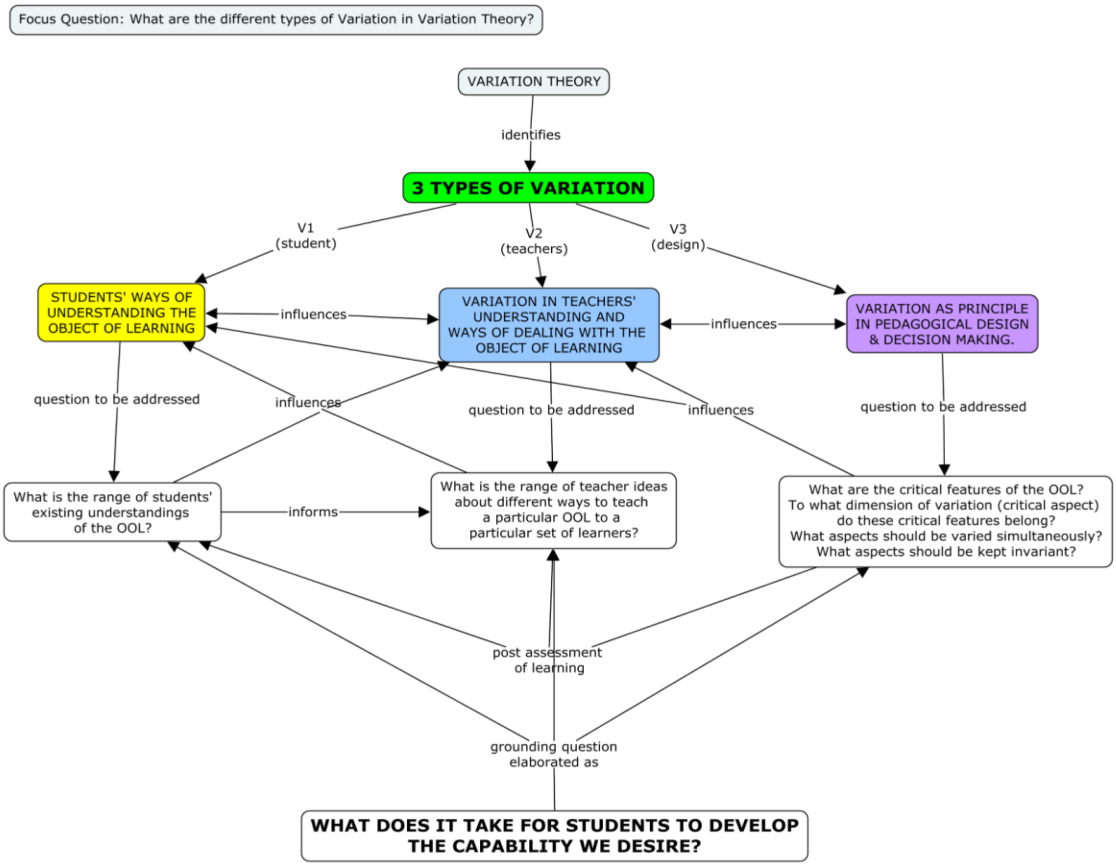


**Figure 3:** The three types of variation within variation theory **(Khan, n.d.)**

In mathematics education, elements of the VT framework can provide practical guidance for teachers in designing mathematical lessons/tasks and can enhance the mathematical understanding of the learners (Handy, 2021; Watson & Mason, 2006). Jing et al. (2017) suggested that only a handful of studies are available on the effects of variation theory when used to inform teaching on students' outcomes. Donovan et al. (1999) pointed out that in order to develop competence in a specific area of inquiry, students must understand facts and ideas in the context of a theoretical basis, and the knowledge must be organized in ways that facilitate recovery and application. Our attempt is to provide mathematics teaching and learning opportunities that can help learners experience mathematics.

In the traditional mathematics classroom, students memorise a formula or algorithm, work through problems individually or in groups, or they take a test to demonstrate their understanding, thus helping teachers to evaluate them. The question here is, how can we empower learners with meaningful experiences in mathematics? In the educational context, the learning experience is the interactions that connect a learner with the matter being learned (Leung, 2010). Leung further identified three categories of understanding (of mathematical concepts by students) based on research conducted in a primary mathematics classroom in Hong Kong: create/discover mathematical knowledge beyond the present level; shape new mathematical knowledge; and re-shape prior mathematical knowledge. Ethnomathematics is present in the cultural practices of various groups of indigenous people and such practices have helped in preserving their cultural identity (Pradhan et al., 2021). It is important that teachers understand how mathematical knowledge is related to various cultures in classrooms with cultural diversity and how socio-cultural factors influence the academic achievement of students (Haghi et al., 2013). The mathematics practices of the classroom should bring meaning to reignite mathematics knowledge, which can be explored through a multitude of activities and concepts, including fascinating designs/techniques/patterns such as kolams.

## A background on kolams

Kolams are drawn using loose powder-like material (i.e., rice flour, chalk powder, rock powder, ground rice, sand, etc.) that connects and surrounds predesigned dots with lines. The practice of drawing kolams is generally performed on a wet surface so that the design stays for a long time. Usually, the dots are connected by lines to make a pattern, or loops are drawn around dots to create a design. The drawings of kolams originated from the practices of women from Tamilnadu, a ritual performed in the early mornings before sunrise. Typically, rice flour used in making kolams can feed many types of animals like ants, birds, and squirrels, and hence this land art is a symbol of harmonious living with nature. Kolam designs are recursive in nature. They start off as simple motifs and form a complex structure by repeating the subunits. There is a synchronization of yoga in the practice of drawing kolams. The health benefits range from improved blood circulation, meditative effect on the mind, and the posture strengthening the body. Kolams are drawn to channel positive energy to one's homes/offices, and it has a calming effect on the mind and body to prepare and face the hardships in store for the day.

## The potential of kolams in mathematics classrooms

The art of kolams can create opportunities for students to recognize and build the context of mathematics applications outside of mathematics (Chenulu, 2007). Kolams can be used in educational applications of key mathematics concepts that include (but are not limited to) counting, patterns, symmetry, fractions, probability, geometry, graph theory, algebraic thinking, and spatial analysis/awareness.

Kolams provide and introduce a holistic environment that can make connections to students' lived experiences along with focusing on the whole child/person approach. It has the potential to not only incorporate an invitational approach into the mathematics classroom; the lessons learned can expand beyond the classroom as lifelong learning, respecting environment/resources, the cycle of life, and spiritual connections with body, mind, soul, and nature.

## How kolams exemplify VT and EM

We see potential in providing an example of approaching the use of both variation theory and ethnomathematics in mathematics education through the practices of kolam drawing as a learning sequence for teachers. Through an ethnomathematics lens approaching the study of kolams, the mathematical aspects overshadow the cultural aspects. The ethnomathematics perspective observes kolams as a resource, as we can extract and recognize the mathematics portions from the cultural practice to make connections within the classroom (Perlata, 2020). Additionally, we notice the capacity to use kolams in the mathematics classrooms, as a compelling example of partnering variation theory with ethnomathematics.

Kolams are exemplary in joining the perspectives of ethnomathematics and variation theory, and the practice brings forth several mathematical concepts in a constructive interconnected fashion. The mathematical concepts expressed through kolam practices can provide opportunities for variation theory in student learning in making connections with critical discernments and allow for focus on the *internal and external horizons of the object of learning*. The object of learning acquires meaning through its external horizon in variation theory approaches (Lo, 2012), and generally relates to the cultural aspects, but this link is not made within teacher education since there is limited time and experience for most teachers in the exploration of ***external horizons that are truly external/different to their own***.

[Images from Figure 4 through 11 produced by O. Lu and S. Rajasekharan].

**Figure 4: Variation and developmental sequence in holding and dropping the rice flour**

**Figure 5: Variation (rounded and slanted dots) from different methods of holding flour**

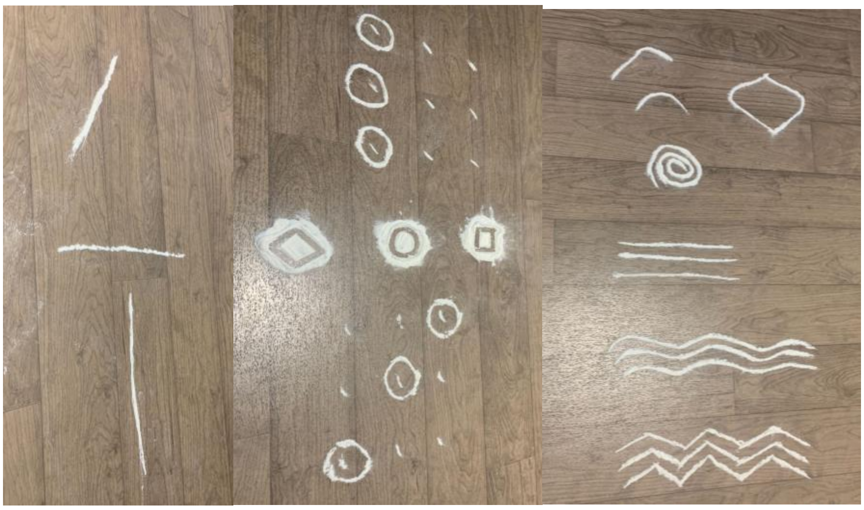

**Figure 6: Sequencing of different dots, simple lines, and curves used in complex Kolam designs**

*Note*. Tracing the design can be used to initiate an interest of students in feeling the resources and partnering with them.

**Figure 7: Dots, circles, and classic shapes/tiles part of traditional Kolam designs**

*Note.* Tracing the designs on the rice flour does not form part of kolam designs. But the authors consider it as a strategy that can be adopted in elementary mathematics classrooms to help create an interest in the subject.

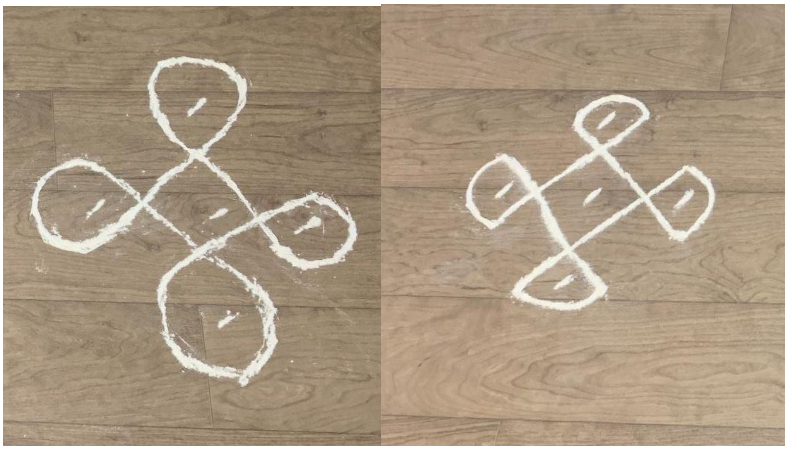

**Figure 8: Variation in the skill level of the 1:3:1 dots Kolam designs**

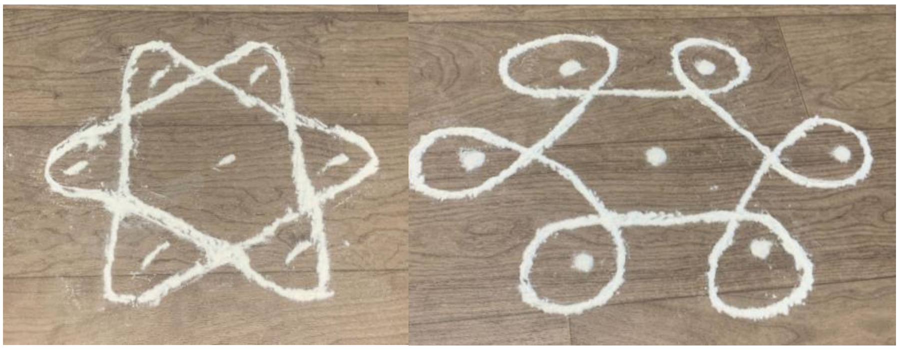

**Figure 9: Variation of beginner patterns created using 2:3:2 Kolam designs**

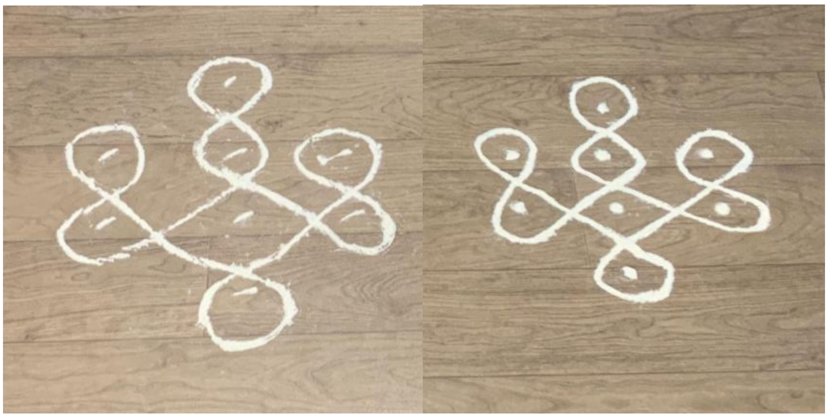

**Figure 10: Variation in skill levels shown in 1:3:3:1, traditional Kolam lamp (wick) design**

## Discernments and drawing explanation VT and EM

Using the framework of variation theory, we would like to probe the pre-lived and post-lived teaching as well as the learning experiences of mathematics educators with their students on particular objects of learning when partnering with culturally responsive instructional designs such as kolam drawings. Specifically, these lived experiences can be brought to the surface with the internal and external horizons as variation theory components, which can directly be indicated with the learning.

From the variation theory framework, the key elements of critical discernments can draw attention to and emphasize the multiple characteristics, movements, techniques, and types of kolam drawings. The critical discernments formed from the kolam drawing practice can be creatively curated with opportunities to relate to mathematical understandings and/or holistic learning concepts, integrated into the classroom. These types of discernments may include choosing the material or style when creating a design. By selecting from various types of flour and sand, the techniques and methods may vary. The process of drawing kolams includes discernments of actions, such as performing body and hand movements to hold or pour the material (i.e., tracing, dropping) (Figure 4). Prior to drawing the selection and method of drawing, the pattern type must be considered with several points of critical discernment. This may include, the number of dots, types of dots (slanted or round), organization of dots (from centre to the outer formation), and types of lines (straight, looped, curved, parallel). Also, critical discernments must be taken into account for the overall design pattern (odd number sequence pattern, connecting dot pattern, etc.), finding a route from starting point to completion, and maintaining the connection with consistent lines to account for each dot with overall sequencing.

In producing the final kolam drawing, critical aesthetic discernments are made in reflection, contemplation, and admiration of the artwork done. Comparison with past drawings and/or the drawings of others provides other opportunities for discernment. These individual and peer comparisons can involve critical discernment regarding learning from different or similar techniques and patterns, reflecting on the pre-planned process in design, and observing the differences in actions that could have been made for improvement.

The critical discernments at play have been displayed throughout this paper. The practices shown in the images of introducing and addressing the critical discernments in the stages of kolam drawings do not have to be followed using the same procedure/organization of step-by-step instruction. In our trials, we only offer an example of some *possible* critical discernments at work to produce kolam designs. Figure 4 illustrates the methods of holding the rice flour, which can be done in three ways, 1) flour between the index finger and thumb, 2) flour pinched by all fingers and dropped with the middle finger, and 3) flour in a fist dropped from the pinky opening of the hand. Figure 5 shows the types of dots and placements of dots are shown in the variation of circular/rounded dots and slanted dots, resulting from the third and first methods of holding (respectively). Figure 6 presents multiple variations and sequencing of different circles, simple lines, and curves. Incorporated together, these produce a kolam design. There are designs of the tracing method (start with material on the surface and then finger draw designs) with classic dot and line circles. Additionally, there are basic types of lines (horizontal, vertical, diagonal) and complex lines (rounded, spiral, zigzag, parallel). Figure 7 demonstrates a variety of shapes created from the tracing method and the line drawing method as well as common kolam shapes, which can also take the form of polypad tiles ("Polypad - Virtual Manipulatives," n.d.). Figures 8–10 showcase the range of complexity in kolam dot line drawings. This first starts with one of the simplest designs with a 1:3:1 design, then with 2:3:2 designs in star and flower patterns, and finally with a more complex design of the traditional lamp kolam pattern. In slowly progressing the aspects that make up a kolam design, critical discernments can draw attention to the areas of improvement and importance for students to learn to make a complex kolam design. Throughout the paper and practice, the experience level of kolam designs is represented in side-by-side images, valuing the beginner and advanced learners and demonstrating the variation in direction and precision based on skill level. The final product of the project is displayed in Figure 11 below. These designs were taken to the natural land surface, honouring the traditional ways of practicing kolam drawing outside in the natural environment. The designs below are the final works of the practice done in previous figures and compiled into one image to demonstrate variation in skill level, symmetry, and the social aspect of drawing kolams.

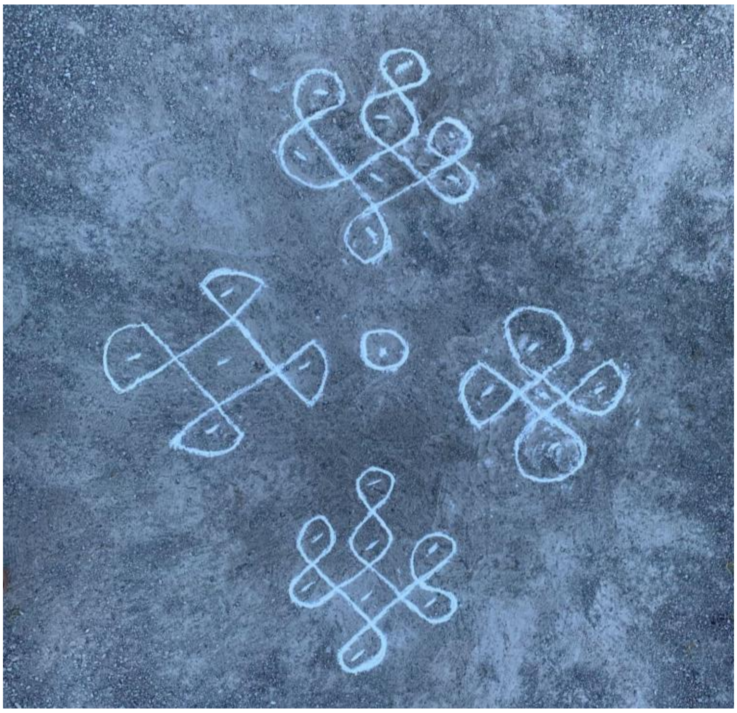

**Figure 11: Holistic realistic practice in demonstrating the final Kolam drawings (variation in comparing skill levels and symmetry)**

## Mathematics for multispecies flourishing (M4MSF)

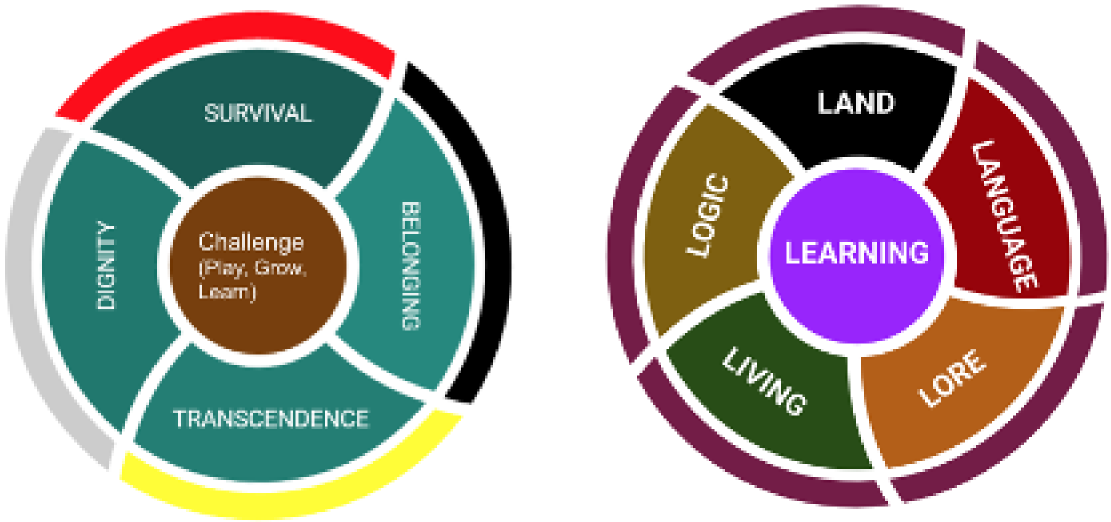


**Figure 12: M4MSF framework (Khan, 2020)**

Mathematics for Multispecies' Flourishing is an ethical framework for the teaching and learning of mathematics that recognizes the right of all species to flourish and encourages respectful partnership between humans and other species. It is framed as extending the frameworks that are attuned to human flourishing (Seligman, 2012; Su, 2020) to the multispecies' world. The latter is understood expansively and in ways consistent with a decolonized and ethical ecology. Mathematical activity or learning experiences are intentionally and explicitly connected with needs for survival, transcendence, belonging, dignity, and challenge through a consideration of land, language, lore (story), living, logic and learning. It entails a period of passionate immersion (see Khan, 2020; Tran et al., 2020).

**Table 1: Rethinking kolams through M4MSF**

| Element | Exemplification from the practice |
|---|---|
| **Survival** | The rice flour in kolams helps in the survival of many species (e.g., ants, birds) including the survival of the art. Here, we connect learning of mathematics with ecological kinship. |
| **Transcendence** | Born of a dot, kolams represent energy or creative power. The women who draw kolams act as creators of positivity and they rise above materialistic thinking and oppression to create auspiciousness in daily lives. |
| **Dignity** | Dignity of kolam as an art form connected to science and mathematics, a combination of tradition and modernity. Teaching mathematics through kolams as a way of telling boys/men that girls/women have been learning mathematics. |
| **Belonging** | Kolams belong to specific cultures in practice, and to the entire universe in philosophy or intention, and in materials used to draw kolams. |
| **Challenge** | In drawing/creating the design, usually without lifting the finger. Getting up early in the morning before sunrise to draw kolams is a challenge. It is a challenge to explore the mathematical aspects of geometric forms, symmetry, number theory, algebra, and other mathematical concepts through kolams (Chenulu, 2007). |

| Element | Exemplification from the practice |
|---|---|
| **Land** | Kolam is a land art or an environmental art that was transferred from remote areas of its origin to modern locations where this art could interact with more audiences and thus widening the scope of this land art (Rahbarnia & Chadha, 2015). |
| **Language** | Kolam is a language of care, sharing (of food), compassion, and universality. |
| **Lore** | Many folk tales exist in Tamil culture to make the kolam drawings mandatory for Tamil households. This can contribute to creativity and imagination in classrooms. |
| **Living** | The rice flour of kolams support many species, and people living in apartments employ kolam artists to draw kolams in front of their buildings and this provides livelihood for the hired hands. |
| **Logic** | Kolam is drawn as an act of charity, to welcome prosperity, and to drive away the evil spirit. Kolams are aesthetic and serve as sources of positive energy. Kolams symbolize the merge of home with the universe, a notion similar to the Hindu concept of 'Vasudhaiva kutumbakam,' which means that the whole world is one family (Tran et al., 2020, Nagarajan, 2018). |
| **Learning** | Measurement is involved in putting the dots and lines. Specific amounts of rice flour to be used for dots, lines. Kolams reflect curiosity, creativity, perseverance, emotional and physical regulation, and confidence and these are vital to learning. Problem solving and innovation are involved in kolam drawings. |

## Conclusion

Partnering ideas from variation theory and ethnomathematics, we believe, has value for teachers of mathematics, education researchers, and curriculum designers. The chief value is a reliable approach to developing critical awarenesses while not succumbing to the tendency to reduce the cultural practice only to its mathematically interesting aspects but to continue to situate both the practice and mathematics as living, evolving aspects of human cultures consistent with mythopoetic (Khan, 2011) and multispecies flourishing (Khan, 2020) framework. Another value is in keeping the cognitive and cultural aspects of practices together in a respectful and appropriate way that is not 'resource' extractive. Kolam drawing provides an accessible, yet sufficiently challenging and mathematically rich, starting point for exemplification and extension of these ideas for teachers and can provide several points of entry for making critical mathematical and cultural discernments while honouring multicultural traditions in mathematics classrooms.

We consider our study important in terms of creating a cultural meaning for students based on the why, what, and how of learning mathematics. Kolam drawing contributes to/supports the skill of discerning teaching strategies/approaches by which a better understanding of mathematics can be created in a cultural setting. Teacher educators need to share a congenial partnership with students in building knowledge, and so together they can develop previous knowledge, perceptions, and creativity for both students and teacher educators. It has been proven by empirical research that higher achievements are the learning outcomes of positive emotions like enjoyment in learning, and lower achievements are connected with boredom and anxiety (Putwain et al., 2020). Shockey (2016) added that certain concepts that may seem unrelated to mathematics, can be used in its mathematics pedagogy under the influence of ethnomathematics. Kolams can be used to explore mathematics that exists beyond the limits/boundaries of academic circles. Mathematics is part of every culture (Ethnomathematics) and how it is practiced by different cultures (for example, the kolam drawings) can be incorporated into the school mathematics curricula. Studies have been conducted to understand the perceptions of mathematics by pre-service teachers and how they struggle to apply their subject knowledge in elementary mathematics classrooms (Burton, 2012). Straats (2006) believed that "If we are serious about understanding mathematics in local contexts of use, we must be willing to ask questions that do not seem mathematical in our own intellectual tradition" (p. 44).

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# Bodymarking: Interpreting embodied experiences of spatial reasoning

Josh Markle[1] and Jo Towers[2]

*Drawing on an enactive hermeneutic theoretical framework, we describe and interpret students' embodied experiences of spatial reasoning in a grade 12 mathematics classroom. In doing so, we employ a novel methodology and tool that we call Bodymarking to create graphic profiles of everyday classroom actions, such as gaze and gesture. Using these profiles, we point to the important role of sensory experience in knowing and doing mathematics.*

*Keywords: Embodiment, gesture, gaze.*

## Introduction

This work sits at the intersection of two critical areas of research in mathematics education: spatial reasoning and embodiment. Spatial reasoning has been identified as integral to both general mathematical capability and the potential for individuals to flourish in life beyond formal mathematics education (PISA, 2021 Mathematics Framework). Research on the body in mathematics education varies widely and includes the constitutive role the body plays in the development of mathematical understanding (Davis et al., 2015), how students experience the body in the mathematics classroom (Roth & Thom, 2009), and how our senses, such as sight and touch, influence how we know and do mathematics (De Freitas & Sinclair, 2014). In this study, we employ a novel process we call *Bodymarking* (Towers et al., 2023), which we use to observe and describe everyday classroom actions, such as gesture and gaze, to offer an interpretation of how students use the body to both sense and make sense in a spatial reasoning activity.

## Theoretical framework

We adopt an enactive hermeneutic theoretical framework (Markle, 2021) to interpret students' embodied experiences of spatial reasoning in the mathematics classroom. This framework is grounded in, on the one hand, the principles of enactivism, which view cognition as a complex phenomenon emerging from interactions between organisms and the environment (Varela et al., 1991), and on the other hand, carnal hermeneutics (Kearney & Treanor, 2015). Because our focus is on sensation in the mathematics classroom, we draw extensively on this interpretive philosophical approach, which views the body as both interpretable and interpretive. Enactivism and carnal hermeneutics share a foundation in phenomenology, in particular Merleau-Ponty's (1945/2012) phenomenology of the lived body. Consequently, we too, are directly informed by Merleau-Ponty in this work.

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## Methodology

Towers et al. (2023) developed a fine-grained tool for mapping classroom action, and use the tool to record and visualize some common ways students and teachers engage each other through intentional movements of the body (e.g., gesture), gazing, and tool use (e.g., writing). Applying the tool yields a color-coded map of the ways students are oriented towards and by each other and their environments in a given lesson. In this study, we apply the process to two 74-minute video recordings, each focused on one of two small groups of students.

## Data

Data generation occurred as part of larger study in two grade 12 classes with a total of 36 participants at a large, western Canadian high school. One of the courses was a calculus class in which 16 participants took part in two sessions. Subsequently, 5 of those participants were invited and agreed to participate in a third and final session. All of the sessions with this first group took place virtually due to COVID-19 restrictions. The second class was a pre-calculus class in which 20 participants took part in three in-person sessions. Both classes were part of the school's International Baccalaureate (IB) program. The focus of this study was on students' embodied experiences of spatial capability in the mathematics classroom. In particular, the aim was to work in spatial ways (e.g., through visualization) on common topics in the secondary classroom (e.g., working with quadratic functions). In this paper, we focus specifically on one lesson in which students worked with parabolas and quadratic equations through spatial activities, including visualization and origami.

## Describing sensation and orientation through Bodymarking

Bodymarking (Towers et al., 2023) is a methodology and fine-grained tool for mapping classroom action. It was originally intended as one of a suite of diagnostic tools developed by a team of researchers (see e.g., McGarvey et al., 2018, 2022), which are designed to indicate and characterize collective action in the mathematics classroom. In the present study, we used the tool to record and visualize some common ways students and teachers engage each other in the classroom, such as intentional movements of the body (e.g., gesture), gazing, and tool use (e.g., writing). Applying the tool yields a color-coded map of the ways students are oriented towards and by each other and their environments in a particular lesson. In addition to identifying collective action in the classroom, we have found it fit for other analyses involving the observation and description of classroom action, such as investigating how metaphors for teaching and learning manifest in the mathematics classroom (Davis et al., 2023).

Preliminary analyses have been conducted on video of mathematics lessons from the Trends in International Mathematics and Science Study (TIMSS), and an example is shown below (Figure 1).

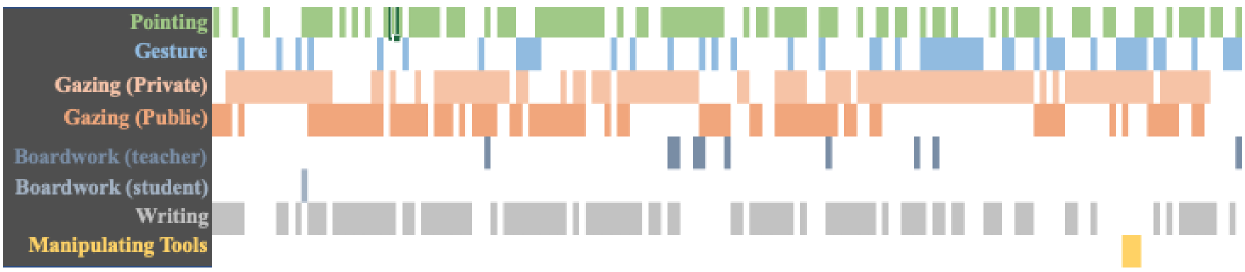


**Figure 1: An example of Bodymarking**

Observations—or, markings, which are indicated by cells of colour—are recorded in a spreadsheet for each of 8 categories at 15-second intervals for the duration of the lesson. The categories, from top to bottom in Figure 1, include Pointing, Gesture, Gazing (Public), Gazing (Private), Boardwork (teacher), Boardwork (student), Writing, and Manipulating Tools. Although these constructs might seem conspicuous for their everydayness, each has been consistently refined and reinterpreted over the course of developing the tool. Moreover, the everyday nature of these criteria speak to the everydayness of our embodied experiences—we spend everyday of our lives in our bodies. For ease of reference, Table 1 provides working definitions for each of the Bodymarking criteria.

**Table 1: Bodymarking criteria**

| Bodymarking Strands | 15 Sec. Intervals | Description of Strands |
|---|---|---|
| **Pointing** | | Using fingers or objects, such as a pencil, to focus attention on aspects of written work, identify key ideas or missing steps, etc. |
| **Gesture** | | Gestures involving the hand while not engaged in pointing; bodily movement, such as modeling distance with outstretched arms |
| **Shared Gaze (Public)** | | Sustained watching of an object of interest in public view, such as when a student stares at a problem written on a chalkboard |
| **Shared Gaze (Private)** | | Sustained watching of an object of interest in private view, such as when a student stares at privately written work |
| **Boardwork (Teacher)** | | Involves addition and/or removal of work by teacher in public view, such as on a chalkboard, whiteboard, or overhead projector, etc. |
| **Boardwork (Student)** | | Involves addition and/or removal of work by student in public view, such as on a chalkboard, whiteboard, or overhead projector, etc. |
| **Writing** | | Addition and/or removal of work in private view, such as on a student's worksheet or notebook |
| **Manipulating Tools** | | Using any sort of tool, including a calculator, working with manipulatives, etc. |

As a tool and process, Bodymarking was not intended to be predictive or even descriptive of any particular pedagogy or mathematical activity. To draw a metaphor from painting, one might argue it is in the tradition of Impressionism, not Realism: its 15-second slivers, like Monet's brushstrokes, convey a sense of movement of things, rather than naturalistic depictions of the things themselves. More concretely, each node of colour signals the "combined intentions of identifying and interpreting – that is 'marking' and 'remarking on' – bodily (inter)action" (Davis et al., 2023, p. 478).

Though the original aim of the Bodymarking tool is thus to give a sense, in tandem with several other diagnostics, of collective action on a classroom scale, we found it furnished us with a lens and grammar for describing and interpreting how the participants in the present study might sense and orient in the mathematics classroom. We opted to scale down the unit of analysis from a classroom-level and focused on two groups of three and four students, respectively, over the course of a 74-minute lesson. We chose this lesson because it was one in which a camera remained focused on each of the groups for the entire period, and because the lesson contained a wide breadth of spatial investigations of topics typically treated as non-spatial (e.g., using the quadratic formula to find tangencies between a line and a parabola).

## Applying the tool

We applied the tool to two 74-minute video recordings of a single lesson, each focused on one of two small groups of students. The video recordings were an important source of data from the larger study on students' embodied experiences of spatial reasoning, so they had already been viewed several times. However, although the video was somewhat familiar, the Bodymarking process created a certain distance between the interpreter and the data. We coded the video without sound, which is an effort to bracket out context. In coding for *Gesture*, for example, the intention is to capture all instances of gesture, regardless of whether or not a particular instance is mathematical or even pedagogical in nature.

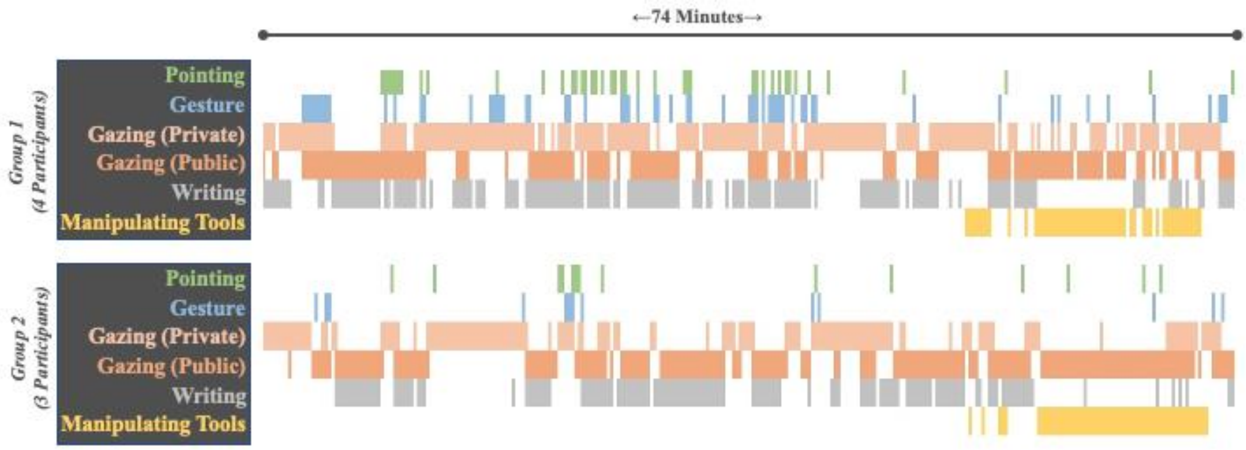


**Figure 2: Mappings from two groups**

Figure 2 shows Bodymarking profiles for the two groups of students. In part because we scaled the unit of analysis from classroom to small group, we removed two of the coding criteria, Boardwork (student) and Boardwork (teacher). This is not to say these may not be important phenomena at this scale, only that neither happened to be present in the resultant mappings for this lesson. Another feature of the Bodymarking process is to code only what is visible in the video recording, so while there were some instances in which the teacher (Markle) was working at the board at the front of the room during the lesson, it was off-screen and thus not accounted for. Moreover, from an enactive hermeneutic perspective, we are most interested in how students touch and are touched by the world around them. We argue this is best captured in the categories that are explicitly about sensation and orientation, namely *Pointing*, *Gesture*, *Gazing*, and *Manipulating Tools*.

Though these two groups were close in proximity, no more than three or four feet apart, their respective mappings bear a stark contrast. Group 1 appears to point (green) and gesture (blue) more frequently, for example. It is intuitive to begin comparing the two groups and searching for correlatives, say in the quality of their written work, but that is not how we employ the Bodymarking tool in this study. Rather, we view the tool as indicative of sense-making in the broadest sense, of how students bring forth a world through sensation and orientation in the mathematics classroom. We allowed the tool to direct our attention to intervals of the lesson that provoked us in some way. We then returned to those intervals in the video recordings and sought to describe and interpret them from an enactive hermeneutic perspective. Figure 3 depicts three such intervals on the Bodymarking maps of the two groups.

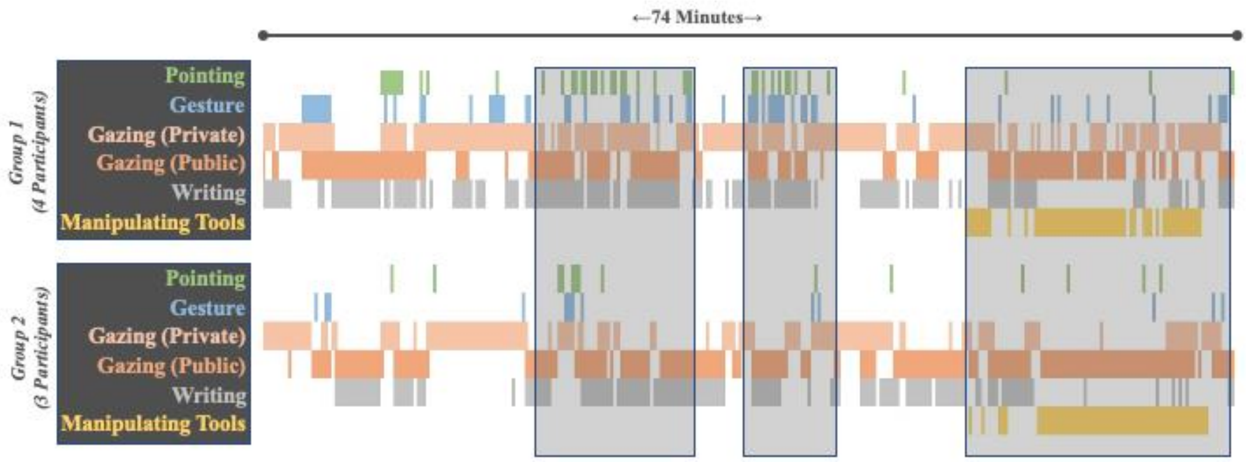


**Figure 3: Intervals of classroom action**

These intervals (shaded regions) stood out for how they reflected distinct cadences of classroom action, in particular through gesture (light blue) and gazing (light and dark brown). Blocks (a) and (b) seem to depict frenetic activity: in both groups, particularly group 1, pointing and other gestures are used frequently, and gaze is in constant flux, private in one moment and public the next. Block (c) captures a different sort of rhythm. In both groups, movement is oriented around tool use, but there is a divergence as well: gaze in the first group continues to oscillate between public and private, while in the second group the gazes seem more fixed and stable. With these insights in mind, we returned to these intervals in the video to see if the Bodymarking process aided us in describing how students experienced spatial reasoning in the classroom.

## Movement: Pointing and gesture

The Bodymarking process focuses the interpreter on specific movements of the body in the classroom. In the tool, pointing is parsed out from the broader category of gesture. Madison (1988) refers to hermeneutic inquiry as a "method for choosing appropriately" (p. 176), and the choice to focus on pointing in particular is deliberate. One reason is connected to the nature of pointing: when we point, we mean to orient ourselves or others (e.g., pointing to a desired location on a map or when one "points the way" to another). Contrary to other gestures, which may or may not be intentional or directed toward another, pointing is a solicitation, a movement that "sketches out the first sign of an intentional object" and "indicates…specific sensible points in the world and invites me to join" (Merleau-Ponty, 1945/2012, p. 191). Pointing is a way we can reach out and touch the world, and solicit others to join us there, but it is also a response to a call, a means by which we ourselves are oriented. As Kearney (2015) noted, we are always "solicited by the flesh of the world before we read ourselves back into it" (p. 45).

But pointing is not the only means by which we bring forth a world. The broader category of Gesture captures the myriad ways we move with intention in the classroom. This can include, for example, raising one's hand to ask a question or using one's arms to measure a distance. Figure 4 highlights two intervals from the lesson in which some form of bodily movement was prominent, whether it be through pointing, gesture, or manipulating tools, such as origami paper.

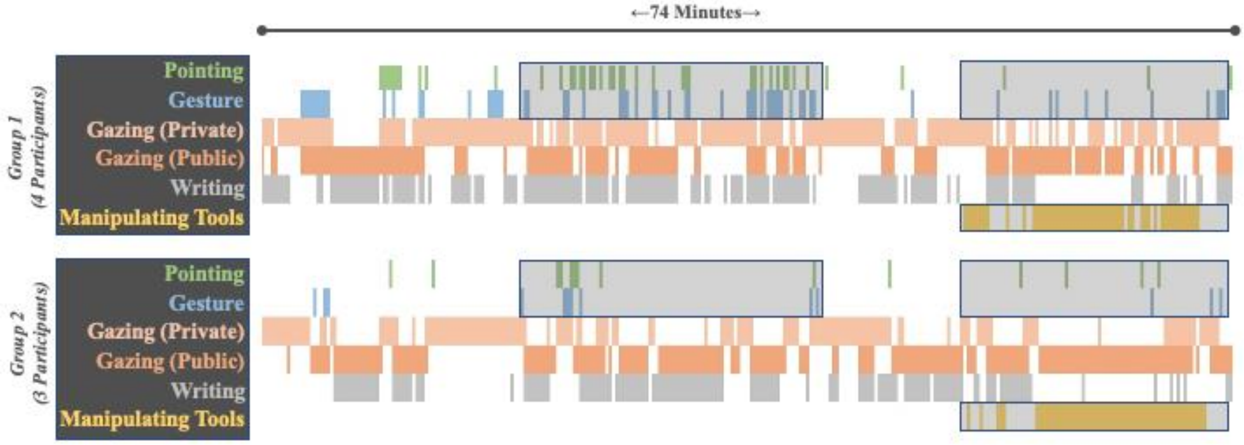


**Figure 4: Pointing, gesture, and manipulating tools**

We have highlighted manipulating tools in this mapping because of something that stood out to us on reviewing the video segment after completing the Bodymarking process. We noticed several instances in which the members of group 1 used touch to explore each other's work. The task involved a simple origami construction, in which an edge is folded onto a point. Doing so repeatedly yields a parabola formed by all of the crease lines. In fact, the provocation in Hull (2013) is: "Doing this origami fold is equivalent to solving a quadratic equation" (p. 49). Figure 5 shows one of these instances.


**Figure 5: Paper folding and movement**

After carefully making some folds (panel a), one student calls attention to their origami paper, framing a crease line between their thumbs as their peers look on (panel b). Three of the students then return their gazes to their own origami paper, but only two of them begin to fold again. The third student instead reaches out across the table and pinches the first student's folded paper between their thumb and index finger (panel c). With their other hand, they tentatively bend their own piece of paper, their gaze fixed on it. After this confluence of gaze and touch, the four students then turn with renewed purpose to their individual folding (panel d).

Two phenomena of interest emerged from reviewing this segment in light of Bodymarking. One is the critical role of touch in spatial reasoning. Although the instructions for folding were diagrammed and presented on the whiteboard at the front of the classroom, and I (Markle) demonstrated the fold in real-time with my own paper, this group of students demonstrates the importance of tactile experience in visualizing and executing a spatial task. In their case study of blind students in mathematics, Figueiras and Arcavi (2014) noted that touch has the potential to be "very fruitful for connections between global and local properties of mathematical objects, as well as for emphasizing properties and processes of reasoning" (p. 131). Though provocative, their analysis often treats touch as a discrete source of sense data rather than a diacritical phenomenon of the entire body. This leads us to a second, more surprising insight that emerged from the confluence of touch and gaze depicted in panel c. In this frame, a student is touching their own, partially folded paper with their left hand, and joining their peer at a sensible point, to paraphrase Merleau-Ponty, with the touch of their right hand. But the way this student's gaze shifted back to their own paper during this exchange struck us, and it is something we did not notice until we applied the Bodymarking process to the data. We take up the enigma of gaze in the next section.

## Movement: Gazing

In this work we delineate between what we call *public* gaze and *private* gaze. A public gaze is sustained watching of a publicly accessible object. For example, during whole-class instruction, students may gaze at the teacher or whiteboard at the front of the classroom. A private gaze is sustained watching of an object intended for private viewing, such as when a student gazes at their notebook. Coding for gaze in in this way involves determining which type of gaze is most prominent: are most individuals looking to an object intended to be publicly accessible over the 15-second interval or are most looking to their own private works? Intervals that contain significant amounts of both public and private gazing are dual-coded. Figure 6 highlights intervals of interest from the lesson with respect to gaze.

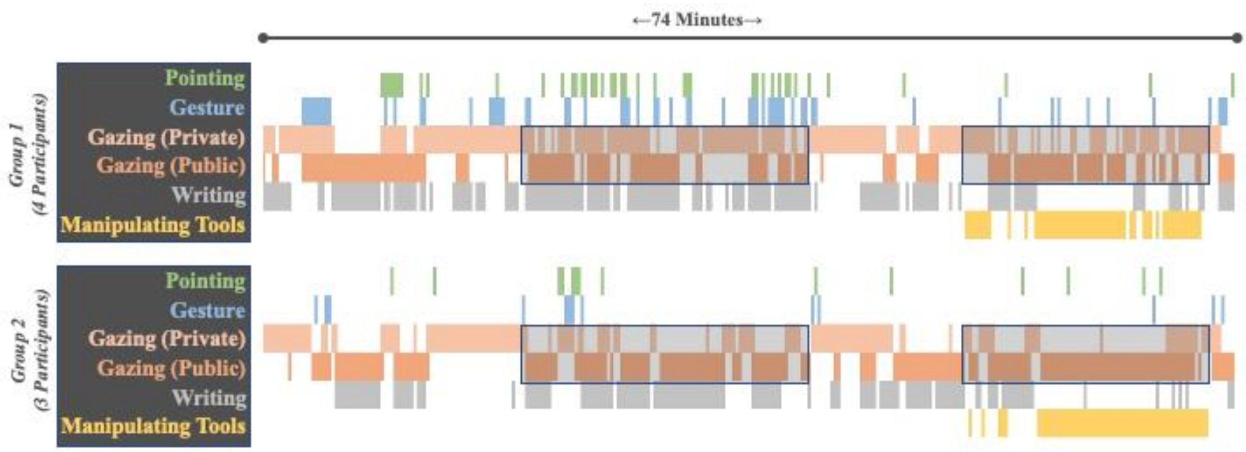


**Figure 6: Intervals of gaze**

From a practical perspective, Bodymarking's delineation between public and private gazing yields a proxy for the nature of classroom action in a given interval. It approximates an answer to the following question: Are individuals mostly oriented by and toward each other or themselves? Coding in this way allows the tool to point the interpreter toward different cadences of everyday action in the classroom. On one level, we argue the distinct cadences of gaze depicted in the highlighted blocks in Figure 6 reflect distinguishable cadences of classroom interaction. Group 1's (top) pattern seems frenetic, while Group 2's (bottom) appears more stable, with long periods of uninterrupted public gazing. This is not to say one pattern of gaze is preferable to another, only that these groups had established their own interactional style over the course of working together in this class. In this sense, following Nemirovsky and Ferrara (2009), gaze can be seen as a form of bodily activity that plays "a part in a given conversational turn or transaction" (p. 162). Indeed, the video segments associated with the highlighted intervals reveal two very different but similarly effective—at least in terms of solving the problem at hand—patterns of interaction. For example, the members of Group 1 frequently check in on each other's work with a glance before returning to their own work, while the members of Group 2 sustain their collective focus on a single group member's work for longer intervals. In this sense, the distinction between private and public gaze is well-defined.

Another moment of interest emerged toward the end of the lesson, and again, questions emerged from the Bodymarking process that did not occur to us to pose during previous viewings. This moment occurred just after the last highlighted blocks of gazing in Figure 6 during a final visualization exercise. Having solved for the tangency between the line and parabola in the first segment of the lesson, then explored the spatial properties of parabolas through paper folding in the next, I (Markle) asked students to visualize the parabola formed between a point (focus) and a line (directrix). Next, I asked them to slowly move the focus up and down, and to visualize what they saw happening to the resultant parabola. Figure 7 depicts group 1 working through the visualization exercise.


**Figure 7: Using origami to visualize a parabola**

Two provocations emerged from this moment, neither of which we had fully acknowledged in our initial viewings of the video. The first had to do with one student's gestures, highlighted in the boxes in Figure 7. As Markle described moving the focus away from the directrix, this student's hands appeared to reflexively open, describing the movement of the parabola as it became increasingly shallow as the focus moved away from the directrix. Though we had noticed this gesture in previous viewings, the Bodymarking process focused our attention to the way this particular student's gaze coupled with the gesture. In this case, we argue, the student's gaze is neither wholly public nor private: the student seems to be looking off into the room, possibly at Markle as he leads the lesson, but is also taking part in a group visualization, which clearly has private aspects (including not only what the student ostensibly sees in their visualization, but their gesture as well). This called our attention to the gazes of the other members of the group, all of whom have their eyes closed. Coding this through the Bodymarking process was difficult. From a practical perspective, these students were not gazing at all—their eyes were closed. But they were engaging in a visualization task in which we all intended to see the same thing. We argue that this is indeed a gaze of a kind, one in which the "seer does not disappear in the visible or vice versa but…forms part of the visible and is in communication with it" (Moran, 2015, p. 230).

## Concluding remarks

Kearney (2015) wrote that the task of carnal hermeneutics is to "revisit the deep and inextricable link between sensation and interpretation" (p. 17). The Bodymarking process provided one means of opening that link to question. It also helped to underscore the potential of reimagining of how we view the body in the mathematics classroom. The importance of gesture in learning has been established (Novack & Goldin-Meadow, 2015), and we see students in the lessons described above using gesture in a variety of practical ways. However, we take a wider view of physical movement in the classroom to foreground the ways in which our senses, through the movements of our bodies, enlist each other in bringing forth worlds of meaning. Moreover, we see through our analysis not only the ways in which a body's senses are entwined, but the ways bodies are entwined through the senses, as in the paper-folding episode described above. This leads us to suggest the importance of recognizing the role of sensation and orientation as ways of knowing and doing mathematics in the classroom, and Bodymarking as a potential means of doing so.

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# Some mathematical models in individual and team ball games

Bienvenu Rajaonson[1]

*This paper sheds light on the transition from practice to theory regarding ball games. To do so, it reviews some patterns experienced in the practice of volleyball. Then, to move from practice to theory, the use of simple first-degree equations related to two parameters has been carried out. The first one is the number of players and the second one is the number of available balls. These equations demonstrate the potential of the algorithms created for their application in the learning and improvement of players' skills. Moreover, these equations open opportunities for their application in other individual as well as collective sports with ball. It then discusses how this applies to the interdisciplinary field of mathematics and physical education. Finally, the modeling performed by this article is yet another way of raising awareness and advancing research on sports and their integration into society.*

*Keywords: Ball game, equation, models, mathematics education, continuous and discontinuous circuits.*

## Introduction

According to the Visual Dictionary, "ball sports are individual, or team sports played by throwing or hitting a solid or air-filled sphere," QA International (2009). Games with a ball are usually a competition between two individuals or collective opponents. They consist in scoring points. To do this, there is a repeated alternation of offensive and defensive actions until the scoring phase is reached or not.

Such ball sports may include games such as basketball, volleyball, tennis, pickle ball, soccer, rugby, and hockey, to mention just a few. Each of these sports involves and requires basic technical movements, which normally conform to the principles and rules of the discipline (Teodorescu, 2013). As a result, the level of competition is and will be more and more complex and challenging. In this regard, apart from the physical and psychological qualities, it is essential to strengthen the players' technical and tactical skills towards less predictable actions. Indeed, it is a game of possession of the ball. On one hand, possessing the ball corresponds to an offensive action with the possibility of scoring a point. On the other hand, the defensive action seeks to dispossess the ball from the opponent (Teodorescu, 2013).

The purpose of this article is to show that training schemes in volleyball have mathematical explanations. They are no exception to the rule. Then, once modelled, the latter can be extended to other individual and team ball games. On the one hand, they provide a better understanding of each of these games. On the other hand, their use facilitates the technical and tactical training and practice of both individual and collective team players. In this sense, they offer an opportunity to learn more about ball sports and mathematics at school as part of an interdisciplinary physical education and sports program.

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As the level of competition continues to evolve, many technical, tactical, and strategic requirements pose a constant challenge, because they are increasingly complex. As a result, there is greater emphasis on "scientific expertise" to support a "race for performance" Delalandre (2010). In other words, ball games are confronted with a systemic problem to which it is important to provide both theoretical and practical answers.

In this perspective, Teodorescu (2013) thinks that it is necessary to approach the problem in a comprehensive way. Because, according to him, “technique is the primary means by which tactical tasks are carried out” (p. 2). So, there is no separation between individual techniques and collective tactics. In other terms, the first could not exist without the other and vice versa. In this sense, Teodorescu adds that, as their main objective, the tactical schemes and technical exercises developed for learning and improvement must tend towards their modeling. Thus, Teodorescu thinks that “for the complete practice of the game, we consider that forms of organization of individual and collective actions are necessary, e.g., tactics for all phases of attack and defense, as well as exercises for learning and perfecting specific tactics not only for these phases, but also for the correlation between phases” (p. 57). Gréhaigne and Nadeau (2015) add another perspective claiming that the spatiotemporal parameters are the most important factors in team sports. They said, “distance measurement can be transformed into time measurement” (p. 80). As such, these parameters may be understood as an interval relationship between players, either with teammates or with opponents. In other words, Gréhaigne and Godbout (2014) interpret that as the translation of an unfolding of an offensive or defensive action according to the variation of direction and distance between the players at a given moment. For them, therefore, modeling is based on the “analysis of the dynamics of the game” (p. 97). Parlebas (2005) argues that “it is possible and important to model mathematically by stable configurations, invariants that summarize the operating systems of the game under consideration. We call these operating systems universals” (p. 15). In fact, once formalized, these “universals” are likely to find applications in other sports with balls. Among other things, Parlebas (1985) has highlighted in his research work the modeling of changes in the role of players based on interactions parameters (offensive and defensive) using the theory of graphs and that of the “scoring system e.g., in volleyball” (p. 37).

The theoretical statements cited above are therefore the main references for understanding the interest and complexity of ball sports. Each of them showed the importance of the modeling process in the organization and practices of these sports. They also showed whether the work proposed was different and whether it could advance research in the field of ball sports. For these reasons, they helped considerably in validating and orienting our methodological approach.

## Methodology

This work is based on several years' experience of playing, competing, and studying volleyball. The author was a national team player at university and at national level. He also coached the men's and women's teams at the University of Antananarivo in Madagascar. These teams both won national championships and gold medals at the 1978 African University Continental Championship. This enabled them to take part in the 1979 Universiade in Mexico City, representing the continent. The men’s team placed 12th out of 24 participants.

The study was conducted in three distinct parts. Section A was devoted to the identification of systemic problems in the context of volleyball training. Then, section B focused on the repertoire of measures taken to both overcome the constraints encountered and seek the achievement of the learning and the development of predetermined objectives. Finally, section C was a direct consequence of the second phase and was dedicated to the mathematical modeling of the measures taken. This last part would allow the generalization of the models for an application to other sports with ball.

## A. The context of the practice

The conditions of practice of volleyball training, in which the team concerned was confronted, were considered systemic and were presented as follows:

- The number of players present at the training varied in each session because it depended on the availability of the players.
- The number of balls also varied, due to various causes such as punctured, badly inflated, worn out, or the cost of new equipment.
- The use of the gymnasium was shared with other sports disciplines. Also, we often were obliged to play in the outside fields without fences, which caused considerable time spent on the recollection of the balls. The training duration was limited.

In short, to remain competitive, the players' training management had to adapt to these recurrent problems and look for appropriate solutions to each session.

Thus, the first aspect for our approach was to solve the time lost to collect back the balls. In other words, we had to optimize the use of the available training time. To this end, it was necessary to integrate into the learning and improvement exercises a circuit for recovering and putting the balls back into play.

The second aspect was to solve the variation in the number of balls available. This was done to ensure that during each session, a player can learn or improve the game through a high number of ball touches and experiences in terms of game situations.

The third aspect was to resolve the variation in the number of players present. In fact, based on the technical and tactical deficiencies or strengths of each player and/or of the whole team, it was mandatory to proceed with various exercises including those personalized and others that tackled collective tactical approaches.

## B. The context from practice to theory

The transition from this observation of practice constraints to a conceptual generalization of their solution was the cornerstone of the work. While solving the specific case in volleyball game, the idea was to develop simple mathematical formulations that could also be extended to other ball sports.

## Practice

Considering the observation seen before, the few schemes given as examples below were developed to support the process. At the same time, they mainstreamed the reduction of the time spent for the recovery of the balls.

- When there was a limited number of balls, the training could be done by dividing the number of players present into two or more groups.
- When there was enough number of balls for the players present, there were possibilities to do individual, two and three-line groups.
- The practices were planned to take place either by workshop rotation or by separate exercises.

The transition from practice to theory took into consideration the two key parameters mentioned above, namely, the number of players and the number of balls.

## Theory

The two parameters that come with the process are therefore numbers. The idea was then to see the possibility of putting them into an equation. Depending on the case, the plan was to organize the session with practical exercises diversified according to the game strategy by making simulations either from the number of balls available or with the number of players present.

Moreover, a discontinuous ball circuit associated with continuous ball circuit models had been introduced in the selected models used in a volleyball practice training system. They were supposed to diversify the form of exercises and increase practitioner’s skills efficiency in anticipating unexpected scenarios in attack as well as in defense positions.

## C. Modeling towards a generalization of the concept

Mathematical modeling was the final phase of our approach. It served to optimize practices whatever the number of balls available and the number of players present at the session. This consisted in putting the key parameters - the number of balls and the number of players - into an equation form.

So, $x$ was named as the number of balls and $y$ as the number of players. They should be put in the form of a first-degree equation and would take the shape of a mathematical model applied to the training of volleyball. As many of the volleyball training schemes are the same as those encountered in other sports with balls, their applicability should follow a case-by-case observation process.

## Results

Hereafter are the modeling results obtained from the methodological approach. They are presented in two sections as follows. Section A is dedicated to presenting the three mathematical models which were developed. Then, section B is devoted to the introduction of the continuous and discontinuous concept circuits of the ball.

Section A. The three equations developed. Here are few schemes used to train players in volleyball.

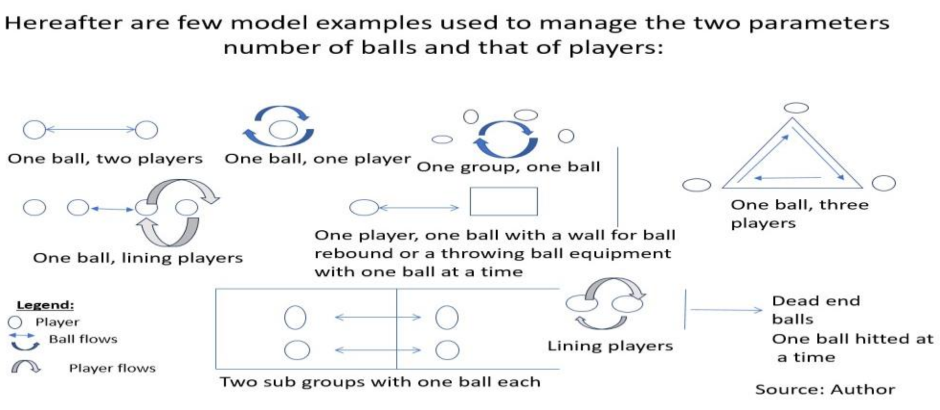


**Figure 1: Some selected models used in volleyball practice**

The examples shown in Figure 1 include both individual and collective practices. The correlation between the two key parameters - number of balls and number of players - is well highlighted. Thus, for example, there is the case of one ball for one player. Then, there are cases of one ball for two players or three players. There is also the case of a ball for one or two groups of players. The arrangements also vary, as they are either in line or in the face-to-face players scheme or in the shape of a triangle. Finally, there is the case of one ball hit at a time, which needs to be picked up back to the practice session. In principle, these examples may or may not be used directly in the game space, depending on the specific content of the training session.

## The theoretical conception of mathematical modeling

The two parameters, namely *x* number of balls and *y* number of players, are put into three equations according to usual volleyball practices seen above. Then, each of the three identified equations has been shown as follows.

**1. First case: x=y**

This equation means, one ball per player. So, this might be extended infinitely as $x=y=n$, $n \in N^*$.

In practice, in this equation, the limiting factor is the number of available balls. So, the number of ball (s) should be less than or equal to y, so, $x \leq y$.

The equation x=y can also be applied if the team is split up into subgroups where the number of subgroups should be less than or equal to y.

This is used to perform individual skills with specific training. The operative scheme should be a continual practice without interruption, also, to be executed in movement, running form, etc., while keeping the ball during the entire allocated time for practice.

**2. Second case: x = y-1**

This equation corresponds to the model set for one ball per two players. When the number of players is increased from one to two, and the number ball remains 1, the equation becomes as follows: with $x = 1$ then $x = y - 1$.

So, if $y = 2$, then the equation becomes $x = 2 - 1 = 1$.

The ball is circulating back and forth from player 1 to player 2 as shown below.

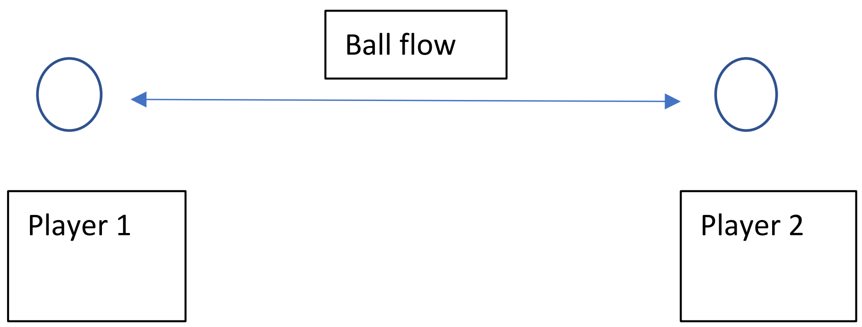


**Figure 2: The lining scheme with two players**

The equation $x = y - 1$ is verified with any ball game's rule. Indeed, using one ball, two teams or two players are competing to score. Based on the above, the total of x balls can be calculated as $x = y/2$ or $2x = y$ where $x = n$, $n \in N^*$.

In other words, if there is only 1 ball available, the only way to perform is to divide the players into two subgroups.

**3. The triangle models: $x = y - 1$**

When a triangle scheme is used, normally, each summit should represent one player. The equation remains as $x = y - 1$ except the number of players becomes $y = 3$. So, from the equation $x = y - 1$, the required number of balls is $x = 3 - 1 = 2$. In fact, it varies from 1 to 2. So, the number of balls is formulated as $1 \le x \le 2$.

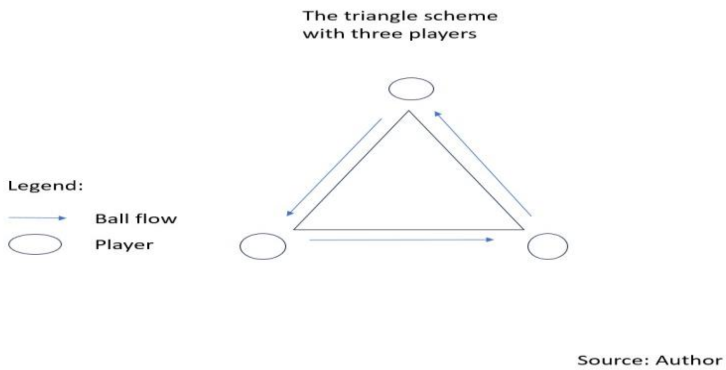


**Figure 3: The triangle model with the use of one ball for three players**

The article will focus more on the model of 3 players with 2 balls in order to highlight the potential offered in ball sports by the introduction of the concept of continuous and discontinuous ball flows, which is developed in the following section.

Section B. The mathematical models enhanced by the introduction of the concept of continuous[2] and discontinuous[3] circuits.

This section shows to what extent the concept of continuous and discontinuous circuits is introduced in each of the three equations presented above.

The first model where $x=y$.

In the case of a one-ball player, the player touches or hits the ball continuously for himself. And then, intermittently, once the ball has left the player's hand(s) or foot(s), he makes an additional side to side gesture.

The second equation where $x = y - 1$

With the case of one ball for two players, where $x = y - 1$, the ball is supposed to be in continuous circuit framework. This means that the ball is circulating back and forth from one player to another one. Then, if a second ball is introduced in the circuit per intermittent time, it is called a discontinuous situation. So, we shall have $x = 2$ and $y = 2 + 1 = 3$, although, the equation should remain the same as stated earlier: $x = y - 1$.

In this case, to determine the moment to go into discontinuous mode, the principles of the interval by introducing the temporal notion have been set.

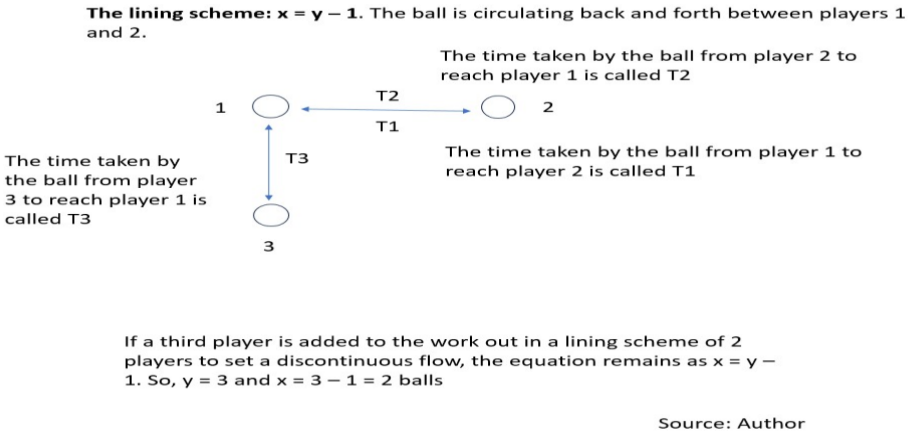


**Figure 4: The lining scheme with 2 players in continuous circuit and a third player added in the work out**[2]

From the above scheme the following interval development has been set.

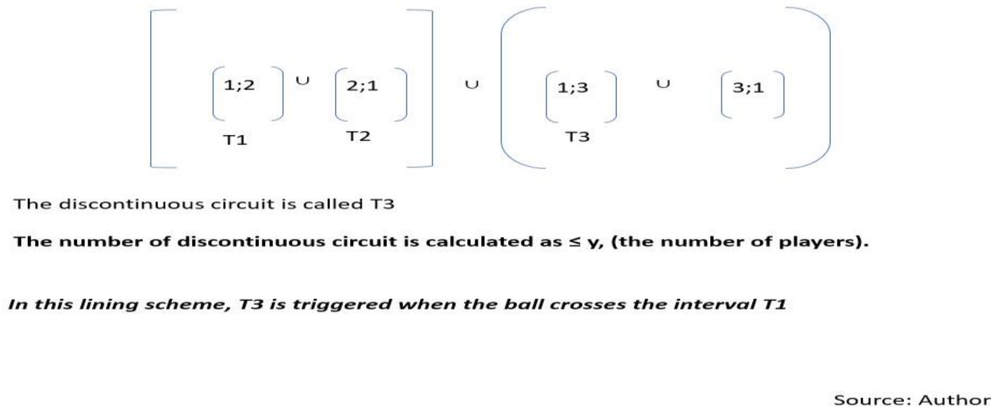


**Figure 5: The interval development of a discontinuous scheme where x = y = 1**

3. The third equation: $x = y - 1$. The triangle scheme with three players is shown below when a fourth player is added as one discontinuous circuit T4 in the work out. So, the equation remains as $x = y - 1$, where $y = 4$. So, $x = 4 - 1 = 3$ balls.

[2] A ball flow is said to be continuous if the exchange between two or more players stops only when one or the other player loses the ball. It is also valid for one-player practices with either a ball launcher whose cessation depends on the choice of the player or the stock of balls, or wall bounces for which the interruption will be caused by the loss of the ball.

[3] A ball flow is said to be discontinuous if the exchange between two or more players does not follow a regular rhythm.

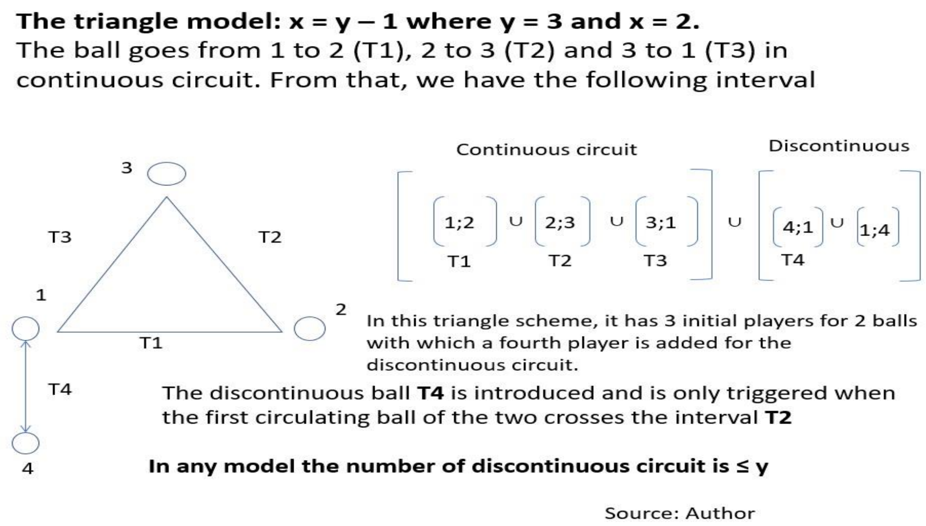


**Figure 6: The triangle model with one discontinuous case**

Then, if this modelling is extended using two triangles' schemes linked with one alternating discontinuous circuit. The result is as follows:

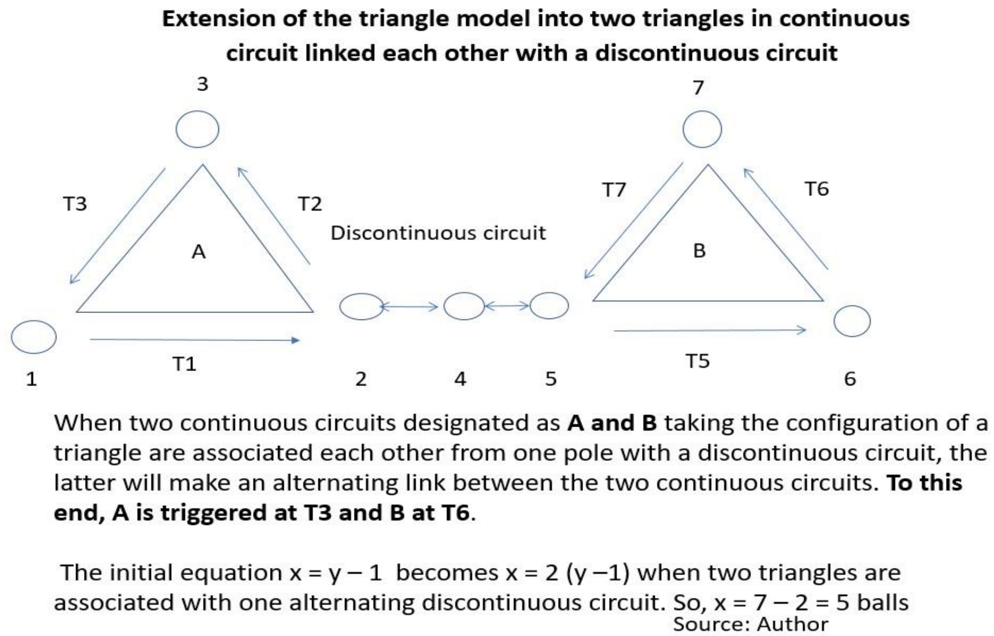


**Figure 7: The triangle model in one discontinuous circuit associated with two continuous circuits**

Therefore, the structure results in an infinite sequence of continuous circuits linked to discontinuous circuits. Each time one more sequence of a continuous circuit associated with a discontinuous circuit is added to the series, the original equation $x = y – a1$ with $a1 = 1$ becomes $x = y – a2$ with $a2 = 2$ … So, theoretically $x = y – an$ where $n \geq 1$ ($n \in N^*$). The introduction of the concept of continuous and discontinuous circuits was intended to show its potential to improve the practice and the technical and tactical performance of players and or of the whole team in ball sports. Especially, the concept enables players to go beyond the acquisition of stereotyped technical and tactical gestures. Thus, by combining continuous with unexpected discontinuous ball circuits, there is an opportunity to create close to reality game situations requiring appropriate anticipation from the players. Indeed, keeping possession of the ball or dispossessing the opponent of the ball with an aim at any cost to scoring is the goal of any ball game (Teodorescu, 2013).

## Discussion

The modeling objective has been performed from the use of two key parameters, which are the number of balls and the number of players. As shown before, these two parameters were put in the form of an equation of first degree. Thus, either the number of balls or the number of players could be put as an unknown parameter.

Regarding the first model $x = y$, where $y = 1$ (Figure 2), if the number of balls allows it, individual exercises can always be done. Dribbling, shooting at the basket, shooting at the goal, for example, are all covered by the model. The use of ball stocks is justified here, as well as a plan for the recovery of the balls at the end of the shooting series. Their number is included in the overall ball count.

Concerning the second model, $x = y - 1$ where $y = 2$ (Figure 3), it models the structure and rules of any sport with balls. In other words, a given ball game takes place on a playing field and pits two entities against each other with a single ball. Thus, the ball used for the restart in case of loss of the ball game is only a practical modality to optimize the practice time. In short, the ball that is used for motor acquisition or confrontation is the only necessary ball that circulates between the opponents. Moreover, this model also serves to justify the use of a wall as a rebound, the latter being considered as a practice partner. The model also includes all the ball throwing equipment or other diversionary tools the players use.

Finally, the third model, $x = y - 2$ where $y = 3$ (Figure 4), is interesting. Indeed, it can be used according to the case that arises either with $y = 2$ or $y = 3$. The advantage of this last case is especially the optimization of the number of balls touched by unit of time.

The advantage of these three models is the fact that they can be executed according to variable distances between the targets and the players or in a precise space that requires a given technical specialization. Furthermore, they rationalize the use of time, promote better performance monitoring, and improve the general organization of learning and development programs for any ball sport.

In addition, the introduction of the discontinuous and continuous circuit models adds to the interest of using these three equations mentioned before. Thanks to these additional models, the exercises to be given to the practitioners can be more varied and complex. Indeed, they aim to provide adequate technical and or other deficiencies adjustment or to improve performance in a particular sequence of the game.

Ultimately, optimizing the learning of basic techniques and tactics, as well as the quest for individual and collective performance, all now require a good calculation of the number of balls and the number of players involved.

In view of the above, a better understanding of these ball games through these simple equations will benefit physical and sports education from an early age, as well as the pedagogy of mathematics in schools. The study led by Masson (2018) stated that students are more motivated when it comes to mathematics education in conjunction with physical education and sports (pp. 10–20, 48). That of Blanchouin and Pfaff (2019), recommended that teachers can succeed in introducing interdisciplinary connections between mathematics and sports education in school “by multiplying the number of examples" (p. 79). So, the mathematical models developed above offer the possibility of a variety of game settings as part of a school physical education program with ball sports. Furthermore, the literature from GREFEM (2018). emphasizes that the use of experienced contexts encountered, for example, in the province of Quebec, confirms their importance in supporting the teaching of mathematics. Consequently, according to Gréhaigne et al. (2020), these models can play as a "student-centered teaching approach that promotes contextual learning and the use of small-sided games" (p. 231). In this sense, they will enable students to design different game situations themselves, as well as calculating the number of balls and players to put in the various compartments and phases of the game. Finally, these equations can be easily combined with score calculations, performance records and statistics, which are, among other things, commonly used in physical education and sports in conjunction with mathematics.

## Conclusions

This paper presented three mathematical equations, based on volleyball practice patterns, which contribute to a better understanding of ball sports. This knowledge opens opportunities to support the learning of ball sports from an early age and the teaching of mathematics in conjunction with physical education and sport at school. Then, the modelling with the combination of continuous and discontinuous circuits presents its potential to develop unpredictable skills required in forming outstanding player. Indeed, the latter are at the heart, among others, of the spectacle, the excitement in the game and ball sports industry. As stated by Parlebas (2005), "Mathematical modeling offers a breakthrough, a new way of looking at sports games that suggests setting more intimately the legitimate nature of game structures with the meaning and symbolism of the actions they perform" (p. 43).

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## Section 4

# STEM & TEACHER EDUCATION

# The importance of mathematical modeling for learning mathematics: Reflecting on the experience from one course for prospective elementary teachers

Amel Kaouche[1]

*Mathematics represents a bridge between many fields of science. These relationships confirm the important role of mathematics in being able to solve problems (situations) in daily life. This paper is related to my academic research and teaching, which include: talking about mathematical modeling and teaching a modeling course to third-year university students. In general, I share my teaching experience: the difficulties encountered by the students, their successes as well as my successes and my recommendations to present and future teachers of primary school mathematics.*

*Keywords: Problem-solving, mathematical modeling, primary education.*

## Introduction

I have been teaching the Mathematical Modeling course to third-year university students for several years. It is a course that prepares students to use mathematical modeling in their teaching. It also teaches students how to solve problems related to everyday life as well as word problems. The course offers a mixture of both the mathematical background that they have already been exposed to in previous courses by ensuring a good comprehension of different branches of mathematics and the interdisciplinary links between mathematics and other disciplines.

The current program of mathematical training for the future primary school teachers includes five mathematics courses, of which the modeling course is the last.

The Mathematical Modeling course plays an integrative role in the teaching and learning of mathematics. It allows future teachers to carry out mathematical modeling projects to solve complex, concrete problems.

By reviewing, in the context of applications, much of the material seen in previous courses that the future teachers were exposed to in a more theoretical context, this course's objective, among other things, is to be able to solve everyday problems: students learn how to move from the concrete (the written problem) to the abstract (the mathematical model) and back to the concrete (the interpretation of the mathematical solution). Also, be able to solve word problems (school problems).

In this course, students are introduced to the steps and processes involved in transforming a word problem into a mathematical model.

Students learn how to solve everyday problems. To achieve this, we use different mathematical techniques for the same problem. We also build on the knowledge acquired in previous mathematics courses, i.e., arithmetic, algebra, geometry, and statistics.

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Here are some examples seen in this course:

1. Applications in finance: simple interest, compound interest, mortgage, etc.
2. Applications of the theorems of Euclidean geometry in everyday life including the Pythagorean theorem, Thales' theorem, etc.
3. Use of the least common multiple and the greatest common divisor to solve certain concrete problems.
4. Use of logical reasoning in certain concrete problems.
5. Use of certain notions in statistics to solve everyday problems.
6. Use of prime numbers in coding.
7. Applications of the theorems of spherical geometry.
8. Applications in cryptography with concrete problems.

Even if the notions of differential and integral calculus are not part of the program of future teachers, there are optimization calculations that are introduced in a visual way. Examples include solving problems of everyday life related to bound rates, calculating the optimal values of a function (which represent a situation in real life), using the notion of derivative, and calculating the area of a surface using certain methods of numerical integration and the notion of primitive. All this with the aim of making future teachers aware of the complexity of mathematical concepts.

In some problems, mathematical concepts such as graph theory are quite useful for modeling. In other problems, tools such as combinatorics are little known or unknown by the students. Students are led to deal with problems in a specific context which goes a bit further than what they have been exposed to in class.

This article is divided into several sections. We first begin with a bit of history concerning changes linked to the use of problem-solving in the teaching of elementary mathematics. Thereafter, we propose four steps to follow for problem-solving in addition to tips students can use in order to better understand each step. Throughout this explanation, we provide strategies that may be used to solve mathematical problems as well as simple examples to further students' understanding. These strategies are then followed by a review of the university course on mathematical modeling where I share my teaching experience, the difficulties that students have encountered in the course, my challenges as an instructor, and the successes that both students and myself have encountered in the context of the course. Finally, I provide recommendations for teachers and future primary school teachers of mathematics as well as people who teach courses similar to the modelling course I teach.

## A bit of history

Many changes took place during the 1980s regarding the use of problem-solving in elementary mathematics education. Previously, the teaching of mathematics was limited to solving problems presented in the form of statements. Examples of this include solving problems at the end of a chapter on multiplication or one of the four mathematical operations, namely addition, subtraction, multiplication, and division. Students would simply take the numbers presented in these problems and multiply them.

The situation changed after the 1980s. Indeed, teachers began presenting some problem-solving strategies as well as steps to follow in order to solve said mathematical problems. The latter required more than one mathematical operation. However, the statements themselves lacked the information necessary to solve them. Despite these changes, no improvements were seen because students could still not solve the problems.

Mathematical problems should be related to real life and not merely number operations. According to D'ambrosio (2003), it is highly recommended by education professionals to give importance to concepts and processes in the teaching of mathematics while pursuing the development of mathematical skills.

The use of situational problem-solving in elementary mathematics education allows students to understand new mathematical concepts and acquire new skills rather than merely presenting mathematical problems only after teaching a certain subject.

We must increase the ability to solve problems in the teaching of mathematics in elementary school. This process is very important because it increases the possibility of students being capable of applying their mathematical knowledge to solve problems in everyday life.

## Mathematics problem-solving steps

Here are four important steps I use in my course for the problem-solving process, which can be used in primary schools:

Step 1: Understand the problem.
Step 2: Make a plan.
Step 3: Execute the plan.
Step 4: Return to the problem.

Note that communication, justification, and reflection are very important parts of the process of solving mathematical problems.

Here we have presented tips that allow for a better understanding of each step.

## Step 1: Understand the problem

The first step in the problem-solving process is to try to understand the problem. Depending on the grade level of the students, they should read the problem several times or be asked to read it aloud to the class. In the case of beginners, the teacher can read the problem aloud several times. Then, the teacher can ask their students to explain the problem in their own words (using their own language).

The next step is to ask students about the problem data (what they know) and the problem question (what they need to find or look for).

We often see that students encounter challenges during this first step in the process. Indeed, language can be a barrier for many students, especially in a diversified class.

The first challenge is intended for students who have the mathematical problem’s language as their second language, and also for students experiencing language difficulties who understand the mathematical concepts and work on them appropriately. These students require support adapted to their reality. There are a few possible solutions to this challenge. These students can be invited to use concrete materials (sticks, tokens, geometric shapes, etc.) or visual elements to contextualize the mathematical problem. Furthermore, the teacher may encourage the pairing of these students with students who speak the language fluently. Another solution is to encourage these students to answer the questions orally or through drawing.

The second challenge is the complexity of mathematical language, which can have a great influence on the degree of students' understanding. For example, it is more difficult to understand a negative statement than a positive statement. Also, expressions such as *most* or *at least* may prove difficult to understand as would the use of mathematical terms having more than one meaning, especially when their meanings in mathematics are different from their meanings in real life. Examples of this include *difference, figure, total, product, factor,* etc.

Here is a simple example: For some students, the word *total* means *addition,* but this is not always the case in mathematical problems.

**Example 1:** Adam has several bookmarks. He decided to give two to each of his four friends and keep five for himself.

What is the total number of bookmarks that Adam had?

**Note:** Here are some student answers: 2 + 4 + 5 (they added all the numbers in the math problem). While the correct answer is: (2x4) + 5.

In this case, the problem can be related to language (how to identify the word total and thus add 2, 4 and 5), but also to the lack of ability to identify the true mathematical structure of the problem ((2x4) + 5).

In my course, I draw students' attention to the importance of identifying the mathematical structure of the problem and not just getting hung up on language.

In summary, mathematics is not only calculations, factorizations, and developments, it is a language in its own right!

## Step 2: Make a plan

The Ministry of Education, Leisure, and Sports (MELS, 2009) offers several strategies applicable to problem-solving that help deepen the understanding of the mathematical aspects of said problem. In other words, to learn to move from the concrete (the word problem) to the abstract (the mathematical model).

### Strategies Used to Solve Mathematical Problems: Inspired by (Dacey et al., 2014)

Here are some strategies, which can be used by students after they have an idea of the problem to be solved:

- Translate the concrete problem into mathematical language: numbers, equations, formulas, etc. This allows the students to establish a good connection between a situational problem and mathematical language. Afterward, students can solve the mathematical equation in question.
- If possible, simplify the problem: Doing so is a way to help students understand the problem. To do this, students can replace the numbers in the problem with simpler numbers. For example, students may use numbers with fewer digits, replace fractional numbers with integers, etc. Teachers should not drown the problem with complex numbers and large

calculations. Once the problem has been solved, students can return to the original problem with more complex numbers.

- Represent the data of the problem with images, tables, lists, etc.: Students need to be able to gather the data of a problem in the correct manner. They can represent the data of a mathematical problem with drawings, examples, or tables to better visualize the relationships between them.
- Bring out (extract) a regularity if it exists: The study of numerical or non-numerical regularities (shapes, colours, etc.) is very important in learning mathematics, and students learn it at a young age such as in kindergarten when they are exposed to very simple problems and age-appropriate language. An important strategy in solving mathematical problems is to reproduce patterns. To do so, students must first find (understand) the relations and deduce a rule. Then, they can predict the outcome.
- Bring out the information contained in a list, diagram, image, table, etc.: An important step in the problem-solving process is students' ability to interpret a list, diagram, image, table, etc.
- Predicting or estimating, in some mathematical problems, can be a good choice: Trial and error can be good strategies when students are struggling. Estimates help students verify a guess (narrow the range of their guesses). If a guess does not work with a smaller number, they can try a larger number and vice versa.
- Represent the mathematical problem concretely: In some problems, it is essential to simulate a situation to better understand the problem in question. Students can manipulate concrete objects or draw a picture to illustrate the mathematical problem.

A good way to empower students is to present them with mathematical problems that can be solved in more than one way (more than one option). So, if one strategy does not work, students can try another. This allows them to learn that when things do not work on one side, they must look at the other side (we must not lose hope!).

**Example 2:** (inspired by Dacey, 2014)

Here is a problem that can be solved using different strategies.

Rassim has five bags of sweets, with six sweets in each bag. He takes two candies from each bag to give to his brothers.

How many candies does he have left?

**Note:** Students may use different strategies to solve this problem.

Some students use a drawing to represent the problem's data while others will solve the problem without drawing. The details in the design differ from person to person.

Some students will include a lot of detail in their drawings. For example, they may include drawings of bags, candies, and brothers. They may even illustrate the candies taken from each bag and given to the brothers, and they will write a simple mathematical equation using addition: 4 + 4 + 4 + 4 + 4 = 20.

Others may be satisfied with simply drawing the five bags, the six candies, and the candy taken from each of the bags. They will then translate the problem into a mathematical equation.

Again, some will write a simple math equation using multiplication: 5 x 4 = 20. Others will write a complex math equation using both multiplication and subtraction: (5 x 6) – 10 = 20.

Students who choose to solve the problem without drawing will instead write two separate mathematical equations to arrive at a solution. The first equation represents the total number of candied using addition: 6 + 6 + 6 + 6 + 6 = 30. Next, they will subtract the ten candies given to the brothers using the following equation: 30 – 10 = 20.

In any case, many students will make the following error (this is common in primary and even secondary school): 6 + 6 + 6 + 6 + 6 = 30 – 10 = 20 (which means: 6 + 6 + 6 + 6 + 6 = 20).

We notice here that the students have extended the mathematical equation instead of writing a new one, which shows that they did not understand the meaning of equality.

Teachers must address this type of problem as it can impact the student's mathematical understanding for quite some time.

## Step 3: Execute the plan

To explain this step, we'll start with the following example.

**Example 3:** (inspired by Bernier et al., 2016 [Documents reproductibles]; Cléroux et al., 2016) [Documents reproducible]):

An Olympic-size swimming pool is being built. To meet current standards, the pool must be rectangular in shape and have 10 swimming lanes 2.5 m wide. If the pool has a perimeter of 150 m, what are its dimensions and what is the total length of rope required to separate the lanes?

Once you've gone through step 1, which involves understanding the problem (what information you have and what you want to find), you can move on to step 2, which involves drawing up a plan: since the pool is rectangular in shape and we're looking for its dimensions, we'll need to find the width and length of the pool. To find the total length of the rope to separate the lanes, we first need to find the length of the pool, then the number of separators for 10 lanes. Which brings us to step 3: executing our plan. We start by finding the width of the pool, as we have more information about it: Pool width = 2.5 × 10 = 25 m.

Once we have the width of the pool and know also that the problem has given us its perimeter, we can calculate its length. Using the relationship between the perimeter of a rectangle, its width, and its length, we find:

$P = 2 \times (a + b)$

$150 = 2 \times (25 + b)$

$b = 50$ m (where a, b and P are respectively the width, length, and perimeter of the pool)

Now we just need to find the total length of the rope to separate the lanes. Since we now know the length of the pool, all we need to do is find the number of dividers for the lanes. Caution: Since there are 10 lanes, we only need 9 dividers (many students will take 10 dividers instead of 9): Total length of rope = 50 × 9 = 450 m.

A good tip for teachers (and future teachers) is to insist on the importance of leaving traces of students' work (communicating their approach and their reflections and justifying them). To help them remember their thoughts, teachers can encourage students to take notes as they work. In this way, students will establish good work habits, which will be useful during the entirety of the educational journey (Goldenberg et al., 2003).

To avoid situations where the student finds the correct result without understanding the problem in question, the teacher must see the student’s approach to remedy any errors and communicate the correct approach.

It is essential to the practice of mathematics to encourage students to work in small groups or at least in pairs. This way, students can communicate, share their thoughts, deepen their understanding, learn new mathematical concepts, and even recognize their errors and correct them. For example, a student notices that their friend has prepared a table summarizing the written problem, so they decide to make one too (to help them understand the problem better).

Teachers need to emphasize the importance of student explanation and mathematical reasoning for a problem and not just a key answer!

In this way, students will recognize the importance of their learning and the value of mathematics in everyday life.

## Step 4: Go back to the problem

This is the final step in the problem-solving process, but it is not the least because it opens the door for further learning. Unfortunately, students give less importance to this step. The latter will allow them to learn how to move from the abstract (the mathematical model) and back to the concrete (interpretation of the mathematical solution). At the end of solving the problem, students must write a complete sentence and not just a number that represents the result of their mathematical calculation. To encourage the students, the teacher can ask them to write their own problems like those they have already solved. In this way, students will consider the problems to be their own creations! This will arouse a greater curiosity in them.

As a simple example, we can go back to example 3 (step 3), after having gone through the three steps and completed all the calculations. Students are asked to write a complete sentence, clearly answering the questions in the problem: The dimensions of the pool are 25 m by 50 m, and 450 m of rope is needed to separate the lanes.

## A look back at the university course in mathematical modeling

Returning to the mathematical modeling course that I teach to students in their third year of the teacher education program, I want to draw attention to the fact that what has helped me, among other things, to succeed with this course is my mathematical background, my pedagogy also acquired over the years, my experience in secondary, college, and university education. But also, the fact that I have three children attending primary and secondary schools. The continuous monitoring of my children's learning at school, their challenges, successes, questions, obstacles, mistakes, reactions to situations at school, and experiences, has helped me enormously in preparing my course (in order to properly orient my students (our future teachers)).

Here are a few examples:

- After his math test, my son realizes that he didn't take the time to read the last question, so he didn't understand it and his answer was far from the right one. Hence the importance of reminding students to read (and reread) the problem before solving it. So, identify the problem data (the starting point), the arrival point, take the time to understand certain mathematical terms, etc.

- My son arrived home with his math exam paper corrected by his teacher (underlining in red his mistakes and the points he lost). However, he wasn't able to tell me the correct answer for the numbers he missed. Hence the importance of always telling students the correct answer to a problem (and not just pointing out their mistakes).

- While reviewing his math notes, my son comes to me wondering why we can simplify this expression: $((5 \times 2) \div (7\times 2))$ but not this one: $((5 +2) \div (7+2))$! That's why it's so important to take the time to explain why!

- My son comes home angry, telling me he's lost points on his math test even though all his answers are correct. After checking, I realized that he had done a mental calculation and had just written down the last answer without leaving any trace of his work. I then explained to him the importance of communicating his approach and justifying it on the basis of what he had learned at school.

- My son is very disappointed with his math result because according to him, even though he answered all the questions correctly, he lost points. After checking, I realized that he had omitted the unit! I explain to him that his answer, 40, would have much more meaning if the unit, meter, had been added.

- Trying to answer the questions for his maths homework, my son didn't know what to do to compare fractional numbers, decimal numbers, improper fractions, and percentages. So, I explained to him that you have to write them in the same form before making the comparison.

- Among the challenges encountered by many students is the transition from the concrete (the written problem) to the abstract (the mathematical model), or finding the simplest possible mathematical model, for example, the use of the least common multiple or the greatest common divisor in solving certain problems, see the following example:

**Example 4:** (inspired by Bernier et al., 2016 [Documents reproductibles]; Cléroux et al., 2016 [Documents reproductibles])

This summer, Matthew and Franc rented neighboring cottages at the same time for 92 days. Matthew mowed his lawn every 6 days, while Franc did so every 8 days. If Matthew and Franc both mowed their lawns on the day they arrived, how many times did they mow their lawns on the same day?

In this type of problem, it's rare for students to think of solving the problem using the least common multiple of 6 and 8, i.e., LCM (6, 8).

- Another challenge for our students is to go from a written problem to a chain of operations, but also to respect the order of operations in the calculation of this chain of operations, see the following examples:

**Example 5:** (inspired by Bernier et al., 2016 [Documents reproductibles]; Cléroux et al., 2016 [Documents reproductibles])

Solve the following problem using a chain of operations.

Diana has 3 packages of 6 candies each and Lily has 40 candies. Lily gives 6 pieces of candy to her brother. The two friends collect all the remaining candy and divide it equally between them. How many pieces of candy does each of the two friends now have?
In this kind of problem, the student may omit the parentheses or even make a mistake in the order of operations when calculating this chain of operations: $(3 \times 6 + 40 - 6) \div 2$.
The same applies to the following example:

**Example 6:** (inspired by Bernier et al., 2016 [Documents reproductibles]; Cléroux et al., 2016 [Documents reproductibles])

Solve the following problem using a chain of operations.

The organizers of the Halloween Ball sold 100 tickets at $6 each. The host's salary is $250. The cost of renting the hall is $150. How much money do the organizers have left?

In this case, the student may make a mistake in the order of operations for calculating this chain: $100 \times 6 - (250 + 150)$.

To do this, you need to take the time to explain how to make the transition from a written problem to a chain of operations and insist on respecting the order of operations when calculating a chain of operations.

First, I explained the four steps for solving mathematical problems with simple examples to the students in my course so that they could become familiar with their use. I also insisted on the importance of their application in this course and of being able to teach them to their future students. The students greatly appreciated using these steps in their problem-solving. Indeed, with these steps, the students know how to start solving the problem, what to do next, and how to finish. They had to start with simple resolution problems in order to use these steps well and then move on to more complex (but not necessarily more difficult) problems.

Then, I explained the different strategies employed in solving a problem. Students were able to benefit from these strategies in order to solve different problems. In this way, our students will be well prepared to teach these strategies to their own students. Using these strategies as a student can be quite beneficial later as a teacher.

## Some challenges (difficulties)

Here are, amongst others, some difficulties that my students encountered in this course as well as my challenges as a teacher. Students were not accustomed to solving problems in mathematics compared to other mathematics courses: concepts, notions, definitions, theorems, and calculation exercises (mathematical operations) unrelated to everyday life situations. They were not used to going from the concrete (the written problem) to the abstract (the mathematical model) and back to the concrete (the interpretation of the mathematical solution). Also, they were not accustomed to using different mathematical techniques and strategies for the same problem.

Another challenge in this course is that it was necessary to build on the knowledge acquired in previous mathematics courses, namely the arithmetic courses, algebra courses, geometry courses, and statistics courses in order to be able to solve the problems. However, I noticed that there was a lack of depth in the understanding of certain concepts or simply an oversight. It was necessary to make reminders of these concepts. This is, after all, a good thing because students need these reminders to solidify their knowledge as future teachers.

Despite their maturity, the students found that the mathematical language can sometimes be complex and that it sometimes confused their understanding of the problem which allowed them to put themselves in the shoes of a younger student who could be lost with the complexity of this mathematical language.

The language of a mathematical problem, whether it is in French or another language, remains a learning barrier even at the university level for students whose language is a second language and for students with language difficulties even if they understand the mathematical concepts.

One of my challenges in this course is choosing problems that have the right level of comprehension for my students. It was necessary to avoid the absence of difficulty for the students to not get bored, as well as an excess of difficulty which can cause frustration.

## Student successes

Once the students are accustomed to or comfortable with the steps and strategies used for solving mathematical problems and are more aware of the challenges, I suggest that they create problems like the ones already seen. Among the successes of students is to deepen their understanding of certain mathematical concepts as well their usefulness in everyday life. Another success is for students to be able to answer their future students' question: What is mathematics for?

*The biggest achievement for students in this course is being able to create their own problems and then create an oral presentation for the rest of the class. This is their favorite step because they consider it a practical internship in mathematics: Being able to create their own problems, explain them to their colleagues, and discuss together the different strategies to follow to solve them. From my perspective as a teacher, seeing students fly on their own with confidence, joy, pleasure, and hope for their future is my greatest achievement.*

## Conclusion

I would like to emphasize the importance of the modeling course in the training of future teachers (in their university career). The difficulties encountered by my students in this course are only a projection of the difficulties encountered by elementary school students in mathematics lessons. Therefore, finding solutions to improve the learning of mathematics in elementary school begins with providing good training for our future teachers.

Here are some recommendations (advice) to give to teachers and future teachers of elementary mathematics and even to people who teach a course similar to mine:

- Perseverance is a fundamental aspect of learning in general and problem-solving in particular!
- We must consider the differences between each student. Each student thinks and reacts differently to a problem.
- We must teach students not to be discouraged and to try another strategy if the first does not work.
- Find a happy medium! The teacher should choose problems with a challenge appropriate to their students' level of understanding. If the problem lacks difficulty, students may become

bored. Excessive difficult may cause frustration and anxiety in students and should be avoided (Sylwested, 2003; Tomlinson, 2003; Vygotsky, 1986).

- Communication, justification, and reflection are very important in the process of solving mathematical problems. Remind students to always leave traces of their work and to write down their processes and thoughts rather than just noting their key answers.
- If you want to teach a new mathematical concept, it is strongly recommended to start with a simple problem (reduce the degree of difficulty), with numbers of a reasonable size (two-digit numbers rather than three-digit numbers or numbers of one-digit and not two-digit for the younger students depending on their level), and simple calculations (not too lengthy) so as not to bury the concept.
- Encourage students to read the problem several times to better understand it.
- Encourage students to explain the problem in their own language.
- Use drawings, tables, diagrams, lists, etc., to better understand the problem and successfully solve it.
- The problem answer must be a complete sentence and not just a number at the end of the problem-solving. For example, instead of writing 6 as the answer, it is strongly recommended that students write: Lucas ate 6 apples this week.
- Never forget units! A number, 12, at the end of a problem-solving session, does not make much sense. It will be necessary to give more precision (12 pens, 12 oranges, 12 monkeys, 12 meters, 12 litres, etc.).
- Differentiate between different types of time. For example, Syrine states school is at 8:15 am, and I worked today for 8:15 hours.

*Mathematics is present and important everywhere in our lives. Teaching mathematics in schools does not stop with the concepts and formulas to be learned. It is necessary to go beyond these notions to better understand the usefulness of mathematics in our lives and its applications to other fields.*

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# Using modelling to make interdisciplinarity in teaching STEM visible: Implications for teacher education

Dragana Martinovic[1] and Marina Milner-Bolotin[2]

*In this paper, we propose a novel curricular approach for integrating mathematics and sciences and highlight the importance of mathematical modelling and interdisciplinarity for teaching and learning STEM. We also give some ideas on how technology can connect mathematics, arts, and sciences. The main focus is on introducing the Educational Framework for Modelling in STEM (EF4M in STEM) we developed to describe both teacher and student roles in the modelling cycle. This framework and the example of how it could be used in teaching STEM through modelling would be of interest for teacher educators, professional development facilitators, and pre-service and in-service teachers of any STEM subject.*

*Keywords: Modelling, science, mathematics, technology and engineering (STEM) education, educational framework for modelling in STEM (EF4M in STEM), interdisciplinary STEM education.*

## Introduction

In this paper, we discuss possibilities for not only multi-disciplinarity, but also interdisciplinarity in teacher education and challenge a conventional view on Science, Technology, Engineering, and Mathematics (STEM) teacher upbringing. Multi-disciplinarity draws on knowledge from different disciplines while still keeping them separate. Interdisciplinarity, on the other hand, integrates knowledge from different disciplines in order to solve complex problems. Our overarching claim is that the existence of the STEM acronym does not guarantee a coherent and cohesive approach to interdisciplinary STEM teacher education and consequently K-12 STEM education (Martinovic & Milner-Bolotin, 2022). We start by reflecting on recent changes in mathematics and science education at K-16+ levels. Then, we describe how the present educational system is not necessarily conducive to the adoption of authentic and humanistic multi-disciplinary and interdisciplinary approaches to science and mathematics learning (Galili, 2011; Hottecke et al., 2010). The ongoing resurgence of calls to create robust STEM education for the 21st century and to build bridges between the different STEM disciplines (Ben-David Kolikant et al., 2020; Li et al., 2020) indicates that these goals have not been achieved yet.

As mathematics (DM) and physics (MM-B) teacher educators, we believe that it is important to develop approaches for STEM education and STEM teacher education that consider the epistemological and pedagogical commonalities and tensions between these fields. For example, in schools, “mathematics and science have often proceeded along parallel tracks, with mathematics focused on ‘problem solving’ while science has focused on ‘inquiry’” (Li & Schoenfeld, 2019, p. 7).

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Discussing with the students why and how this happens might alleviate anxiety over teaching or learning STEM. Moreover, educating students in STEM by teachers who likely lack the necessary multi-disciplinary content background and have limited knowledge in the history and philosophy of STEM, is also problematic. Consequently, many students perceive STEM as a family of loosely connected subjects. The students rarely have an opportunity to acquire the skills and abilities to traverse the STEM subjects' boundaries. Thus, it is not surprising that despite the ongoing STEM education efforts, the successful and sustainable implementation of authentic STEM pedagogies has stagnated over recent decades (Chachashvili-Bolotin et al., 2021; Chachashvili-Bolotin et al., 2016).

The tensions between teaching different STEM subjects are clearly visible in the current mathematics and science education (Ben-David Kolikant et al., 2020; Martinovic & Milner-Bolotin, 2020; 2022). While in Canada, teacher education varies from province to province, we have observed some common challenges in preparing future STEM teachers, which could be alleviated through collaboration within and between multi-disciplinary teams of educators. We also observed that the expectation that the use of technology will automatically coalesce the STEM disciplines has not fully materialized.

This realization and our experience as long-term mathematics and science educators motivated us to consider modelling as an authentic pedagogical approach that can help glue STEM disciplines in a meaningful and productive way (Martinovic & Milner-Bolotin, 2021). We define modelling "as a cyclical process of generating, testing, and applying knowledge while highlighting the epistemological commonalities and differences between the STEM disciplines" (p. 279). In a modelling cycle, the students generate, test, and apply knowledge that is new for them and sometimes even new for their teachers.

We agree with Hallström and Schönborn's (2019) view that "models and modelling can be used as a basis to foster an integrated and authentic STEM education and STEM literacy" (p. 1). Kertil and Gurel (2016) emphasize that teaching modelling requires more interpretive skills from teachers, which is a challenge that could be addressed through multi-disciplinary collaborations, such as ours. In one of our latest publications (Martinovic & Milner-Bolotin, 2021), we explored the four well-known bodies of knowledge: Kolb's Experiential Learning Cycle (Kolb, 1984); Gardiner's Framework for Epistemic Control (Gardiner, 2020); Model-Based Inquiry Learning (Windschitl et al., 2008), and the framework for teaching modelling (Carlson et al., 2016). As a result, we proposed the Educational Framework for Modelling in STEM (EF4M in STEM), which describes both teacher and student roles in the modelling cycle. We further used this framework to suggest how it could be implemented in pre-service teacher education and in-service teacher professional development. This framework may be helpful in addressing some of the challenges mentioned above. By introducing students and teachers to the process of modelling, we can start building the common STEM language and move beyond the acronym to create authentic, productive, and humanistic STEM learning environments.

## STEM education: What it is and what it isn't

Traditionally, mathematics assumed a service role while engaging students with other STEM subjects (e.g., using examples of mathematical procedures in computer programming; linear and quadratic functions in kinematics; Euclidian geometry in optics; molecular graphs or logarithmic functions in chemistry; projective geometry in drawing area maps in geography, etc.). In that way, mathematics was used but rarely taught or carefully examined in other STEM subjects. For example, science teachers often refrain from discussing mathematics behind the scientific concepts, such as pH level in chemistry, sound level in physics, or earthquake scale in geography in order to simplify science lessons and open them to the students with lower levels of mathematical proficiency (Milner-Bolotin & Zazkis, 2021).

These experiences align with Niss's (1994) description of the societal role of mathematics as the relevance paradox. On one side, mathematics is ever more important and present in all aspects of life, while at the same time remaining largely invisible, staying in the background. Even the scientists in other fields,

> appreciate the mathematics they make use of but simply think of it as a necessary or convenient tool in the service of purposes to which mathematics is of no independent interest. In this respect, mathematics is invisible like the wood that we cannot see because of all the trees (p. 372).

This widespread view of mathematics affects mathematics education as well, justifying it from a utilitarian perspective. Therefore, Williams et al. (2016) ask: "How do we differentiate an interdisciplinary approach to science that brings mathematics in as a tool, from that which brings in mathematics as a generalization of scientific concepts?" (p. 14). Their topical survey of interdisciplinary mathematics education allows these authors to advance the idea that, "interdisciplinary mathematics education offers mathematics to the wider world in the form of added value (e.g., in problem solving), but on the other hand also offers to mathematics the added value of the wider world" (p. 13).

The intricate relationship between mathematics and science is akin to the relationships between theory and application, ideal and real, or imaginary and concrete. Niss et al. (2007) use the term extra-mathematical world to describe the part of the real world to which some concrete problems and ideas apply. For Niss et al., "in any application of mathematics a mathematical model is involved, explicitly or implicitly" (p. 4). During a modelling cycle, one moves between the domains of mathematics and extra-mathematical world iteratively, starting from defining the questions and aspects of the extra-mathematical world one is interested in, then mapping them into the world of mathematics, where initial solutions are made, and then trying them out in the extra-mathematical world, and so on.

Nowadays, other areas are finding their place in the mathematics curriculum (e.g., coding/computer programming; modelling through the use of real-life data; etc.). One noticeable change in the current Ontario mathematics curriculum (Ministry of Education Ontario, 2020) is inclusion of knowledge that may belong to other subjects or is extracurricular; a reversal of what was previously happening with respect of mathematics in other subjects. Despite the widely accepted notion that STEM subjects are closely related, every STEM discipline has its own epistemology, which makes teaching STEM challenging. This is even more challenging if the policy makers start adding more letters to the acronym, such as in STEAM, STEMM, and others (Herranen et al., 2021).

We maintain that labeling something is insufficient to make it a coherent and cohesive concept. Answering the following questions might elucidate the connections between the STEM disciplines and contribute to the design of meaningful STEM learning environments: When we use real-life data in the mathematics class, do we teach mathematics or STEM? When we teach physics and use mathematical concepts to solve problems and do mathematical calculations, do we teach STEM? When we use technological tools to collect, analyze, and model data, do we teach STEM? How do we delineate STEM versus non-STEM? And what does it mean to teach STEAM or STEEM (Herranen et al., 2021)?

## STEM teacher education

## The interplay between the epistemology and pedagogy in STEM education

To study a discipline, one needs to understand its epistemology; to teach a discipline, one also needs to master its pedagogy. Epistemology investigates how we come to know something: What is the difference between knowledge and belief? What is the process of knowledge production? How are disciplinary beliefs and knowledge formed, justified, and validated?

Driver et al. (2000) argue that

> the claim 'to know' science is a statement that one knows not only what a phenomenon is, but also how it relates to other events, why it is important, and how this particular view of the world came to be. Knowing any of these aspects in isolation misses the point (p. 297).

Pedagogy investigates the methods and practice of teaching: How to deliver the curriculum? How do students learn in different environments? How to assess their knowledge and skills to promote learning? How to use various technological tools to support student learning (Ben-David Kolikant et al., 2020)?

Teacher education and professional development need to consider both epistemology and pedagogy in the context of research, practice, and policy. Gardiner's (2020) metacognitive framework for epistemic control (Fig. 1) encourages educators "to go beyond often fixed adversarial critical thinking approaches and to develop an epistemic position based on inclusive collaboration and emergent creativity" (p. 1). This framework was originally developed to facilitate collaborations between students from different epistemological backgrounds. It asks for metacognitive introspection from all collaborators. When learners face a complex problem or work in an interdisciplinary team, they need to move through the states of (a) epistemic awareness, (b) humility, and (c) empathy, in order to reach (d) epistemic control. By understanding what and how members of their discipline know and what the limitations of that knowledge are, one develops epistemic awareness and humility. Seeing that others' ways of knowing are valuable develops epistemic empathy. After one opens up to multiple approaches, perspectives, and practicing multiple ways of knowing, one gains epistemic control.


**Figure 1: Visualization of Gardiner's (2020) metacognitive framework for epistemic control**

## Developing a vision for STEM teacher education

STEM education researchers appreciate the importance of epistemology. For example, Moon and Rundell Singer (2012) suggest replacing the focus on "a content-specific definition of STEM" with "a more epistemic one—the sources, strategies, or practices from which…STEM knowledge comes and…is shared" (p. 32). Then, the educators could invest into delivering "an assemble of practices and processes that transcend disciplinary lines and from which knowledge and learning of a particular kind emerges" (p. 32).

Our vision for STEM teacher education considers modelling as a glue that keeps different STEM disciplines together. As a process, modelling involves aspects that are pertinent to all four STEM fields (e.g., inquiry, quantitative and design thinking, use of technological tools, multiple representations, algorithms, and models). Yet, in mathematics education, modelling may be easily confused with problem-based learning and problem-solving, which might hinder its utility (Bloom, 2015). In science education, also related to modelling, inquiry-based learning is often inadequately taught (Windschitl et al., 2018). Both in schools and in teacher education programs, inquiry-based learning may be driven largely by one's curiosity, superficial and surface understanding of the content, while lacking the "epistemic framing relevant to the discipline" (p. 941). In our decades-long work with future mathematics and science teachers, we have noticed that their beliefs about the nature of STEM disciplines are often taken for granted, creating a space for overlooking how teachers' personal epistemological stances affect their teaching (Hofer, 2001). Since modelling is often associated with dealing with ill-structured problems, like Hofer, we are concerned that university graduates are not well equipped or motivated to solve such problems. Hofer suggests that "education that focuses on the progression of epistemological thinking has the potential for addressing this critical need" (p. 369).

For mathematicians, scientists, and engineers, modelling is a powerful methodology, along with design and experimentation (Ortiz-Revilla et al., 2020). It may be seen as "the most relevant characteristic of the scientific mode of knowledge production" (p. 870), as no modern scientific research is possible without modelling. Educational researchers are full of praises for modelling. Apparently, it leads to remarkable learning gains, especially in underserved student populations and students at risk, or as Lesh et al. (2010) write, "modeling is virtually unparalleled in the successes that it has produced" (p. 283). Lesh and Yoon (2007) highlight how "models & modelling perspectives reject the notion that only a few exceptionally brilliant students are capable of developing significant mathematical concepts unless step-by-step guidance is provided by a teacher" (p. 163). These are sound reasons to include modelling in the mathematics curriculum and expand to STEM education in general.

## Implementing modelling in STEM education: Challenges and opportunities

Modelling that uses mathematical tools is challenging for both students and teachers (Xu et al., 2022). The main issue arises from its reliance on the competences (such as reliably collecting and using data; exploring scientific phenomena, making financial decisions, and computer modelling), that deviate from those that belong to the traditional mathematics curriculum. It is of no surprise then, that in classrooms there were often recorded evident departures from the curriculum expectations involving the modelling process. In addition to challenges regarding the time needed to organize a full modelling cycle and assess it, the teachers are found to experience mathematical, pedagogical, and epistemological challenges (Manouchehri, 2017). Based on Manouchehri's analyses of a 25-hours long professional development with the middle and high school teachers in the US, mathematical challenges included difficulties of identifying variables, what information to keep, when to approximate and use heuristics, and when to use formulas and exact algorithms. Pedagogical challenges included teachers' confusion about short- and long-term outcomes and assessment of modelling, especially since most modelling activities are organized as group work. And finally, epistemological difficulties related to perceived subjectivity of the consideration of what is important and what is not, is the model adequate or needs more refinement, is the background knowledge adequate, and so on.

Lesh and Yoon (2007) highlight how

> the modelling cycles that problem solvers go through generally involve systematically rethinking the nature of givens, goals, and relevant solutions steps - or patterns & relationships that are attributed to surface-level data. Therefore, the most significant things that are being analyzed and transformed (or processed) are students' own ways of thinking about givens and goals – and patterns and regularities that are attributed to (rather than being deduced from) the information that is available. (p. 167).

Similar ideas are emphasized by Carrejo and Marshall (2007), who caution that if not adequately prepared, "many teachers may [rely on] direct instruction methods that do not facilitate conceptual understanding or abstraction [and even] abandon an inquiry-based approach altogether" (p. 48). Pollak (2011) also finds it problematic when teachers are too prescriptive:

> The heart of mathematical modeling, as we have seen, is problem finding before problem solving. So often in mathematics, we say "prove the following theorem" or "solve the following problem." When we start at this point, we are ignoring the fact that finding the theorem or the right problem was a large part of the battle. By emphasizing the problem finding aspect, mathematical modeling brings back to mathematics education that aspect of our subject and greatly reinforces the unity of the total mathematical experience (p. 64).

We conclude then that modelling is not a straightforward process but is one that could be conducted in different ways. Teachers would benefit from having more clarity on *their*, versus *their students'* roles during modelling. The framework that we developed attempts to clarify what teachers need to do prior, during, and after the modelling activity, and when and how teachers should allow students to own the process.

## Introducing educational framework for modelling in STEM

Our Educational Framework for Modelling in STEM (EF4M in STEM) consists of six stages. It focusses both on teachers and students; it is cyclical and iterative and aims at generating working hypotheses through modelling. As the students advance through the stages, their control over the modelling process increases, while the teacher becomes more like a facilitator (Fig. 2).

Four educational approaches lay the foundation for EF4M in STEM (see Table 1) (Martinovic & Milner-Bolotin, 2021). Each one of them centres on a different aspect of learning. First, as discussed earlier, Gardiner's (2020) framework for epistemic control focuses on epistemology and how learners negotiate epistemological differences during their collaborative work. Second, Kolb's Experiential Learning Cycle (1984) sheds light on how students work individually as well as in groups while being engaged in active experimentation. Kolb's cycle also emphasizes the recursive nature of learning: to acquire a deeper understanding and skills learners return to the same concept multiple times going through the sequence of experiencing, reflecting, thinking, and acting stages.

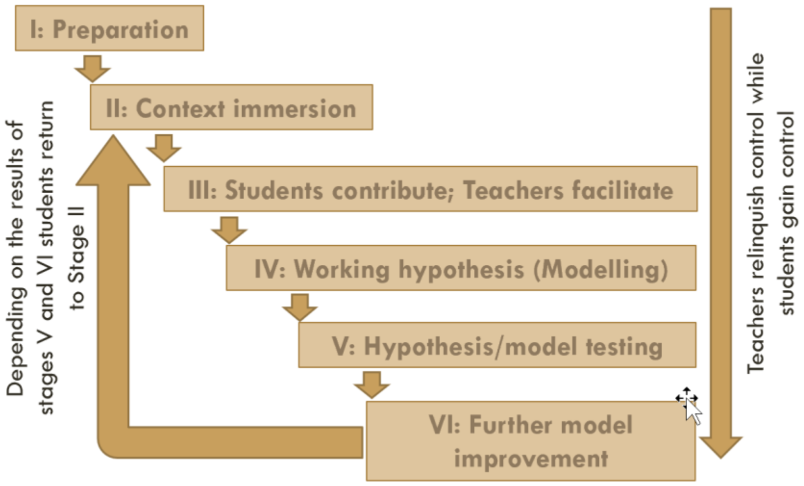


**Figure 2: EF4M in STEM (Martinovic & Milner-Bolotin, 2021)**

Third, Windschitl et al. (2008) suggest a five-step Model-Based Inquiry (MBI) approach in teacher education and professional development, "[t]he goal of [which] is to develop defensible explanations of the way the natural world works" (p. 15). The authors underline the importance of the initial step in inquiry, when a teacher chooses a phenomenon, introduces it to the students, and makes it relevant to students' experiences. Classroom discourse is of ultimate importance for the MBI, as it follows a true scientific approach to inquiry.

Finally, the fourth framework for teaching modelling in elementary grades comes from Carlson et al. (2016). It consists of three phases—posing questions, building solutions, and validating conclusions. Before the first phase, the teacher develops the activity and anticipates potential problems. During the lesson, the teacher organizes student groups, monitors them and regroups them when necessary. After the students complete the modelling activity, the teacher may revisit it to consolidate knowledge or to make relevant curricular connections.

**Table 1: Foundations of EF4M in STEM**

| | Foundational works and their focus | Authors |
|---|---|---|
| 1 | *Gardiner's framework for epistemic control* describes how people from different fields collaborate. | Gardiner (2020) |
| 2 | *Kolb's learning cycle* describes how students learn through hands-on experiences. | Kolb (1984); Morris (2020) |
| 3 | *Model-Based Inquiry (MBI)* informs teachers how to guide students through scientific inquiry. | Windschitl et al. (2008) |
| 4 | *Framework for teaching modelling in elementary grades* provides pedagogical ideas about conducting modelling activities with young students. | Carlson et al. (2016) |

Each foundational theory provided an important insight into the modelling in STEM, allowing us to address learning, teaching (pedagogy), and epistemology relevant for STEM. A more detailed version of EF4M in STEM is given in Martinovic and Milner-Bolotin (2021, p. 293).

## STEM modelling examples

Below, we provide two examples of STEM modelling activities appropriate for middle school, to illustrate how our modelling framework can be implemented in practice. The first example explores the investigation of Newton's laws and specifically the concept of buoyancy and its applications (Fig. 3). The second example examines the concepts of motion through one-dimensional kinematics. These examples could be discussed in middle school or expanded to secondary school classes.

## Example 1: Investigating the force of buoyancy through modelling

The goal of this modelling activity is to explore the concept of buoyancy and its applications in mathematics, art, science, and engineering. After examining the concept through modelling activities, the students will be asked to apply it onto one aspect of everyday life they find interesting and relevant, such as the applications shown in Figure 3.

Stage I: Preparing and building the background knowledge: Every child who has ever stepped a foot into a bathtub, a swimming pool, or an open water, has experienced buoyancy. Yet, understanding buoyancy requires learners to master and combine a number of abstract concepts and be able to describe the relationships between them mathematically.

In the first stage, the teacher prepares materials, resources, tools, technology, and instruments suitable for the planned activities. For example, the teacher prepares containers with water and solutes (e.g., salt), scales, rulers, a collection of objects of different densities, shapes and sizes, and computers with access to simulations software and the Internet. The teacher should consider students' prior mathematical and science knowledge or the new knowledge that they will need to develop to accomplish this modelling activity. To be able to facilitate this activity, the teacher needs to explore different applications of this concept in other fields, such as arts, science, history, etc. (Fig. 3) and consider those that are suitable for the grade level and the curriculum. In addition, the teacher might explore educational research literature on possible student difficulties and misconceptions in understanding buoyancy, force, vectors, linear functions, ratios and proportions, multivariable expressions, etc. The students may be invited to speak about their experiences with floating or sinking, swimming, rowing, or flying and to discuss ways in which to develop understanding of related natural phenomena.

Buoyancy is a force exerted on an object submerged in a fluid (i.e., gas or liquid). The magnitude of this force depends on the density of the fluid ( $\rho_{\text{fl}}$ ), the volume of the fluid ($V_{\text{displ fl}}$) displaced by the submerged object, and on the acceleration of gravity ($g$), as shown in Eq. 1.

$$F_{\text{buoyancy}} = \rho_{\text{fl}} g V_{\text{displ fl}} \qquad (1)$$

The concept of density is often confused with weight, even though they are absolutely different physical concepts. Object's weight is the downward force of gravity (Eq. 2) applied to the object by the Earth. It has both the magnitude and direction, and is measured in Newtons (or pounds). On the other hand, object's density is a scalar quantity that represents the amount of mass in a unit of volume. It is measured in kg/m$^3$ or g/cm$^3$. Dense objects can be heavy or not, depending on their volume, and very heavy objects might not be dense. Thus, a barge can be very heavy, but its average density is less than the density of the surrounding water, thus the barge floats. To understand if an object is going to float or sink, one must compare the magnitude of the gravitational force (Eq. 2) acting on the object downward with the magnitude of the force of buoyancy acting on the object upward (Eq. 1).

$$F_{\text{gravity}} = m_{\text{obj}}g = \rho_{\text{obj}}V_{\text{object}}g \tag{2}$$

Notice, that while the equations describing these forces might look similar, they bear important differences. Thus, in order to predict if the object will sink or float and by how much it will be submerged, one must compare its density with the density of the surrounding fluid. Since the object cannot displace more than its own volume, in order to float, its density must be less than the density of the surrounding fluid.

In STEM classes, teachers might be tempted to provide students with these formulae for the sake of a "pedagogical shortcut," however, there is ample research evidence that demonstrates students struggle understanding these concepts when exposed to them directly. Moreover, solely memorizing the equation describing the force of buoyancy is insufficient to develop a meaningful understanding of the phenomenon and be able to answer conceptual questions (Mazur, 1997). Thus, modelling provides an excellent opportunity for the students to explore this phenomenon and learn how to apply it to understand the world around them.

<u>Stage II:</u> During this stage, the concept of density is examined by experimenting with different concentrations of salt dissolved in water and using different objects whose volume and mass are determined mathematically (Fig. 3).

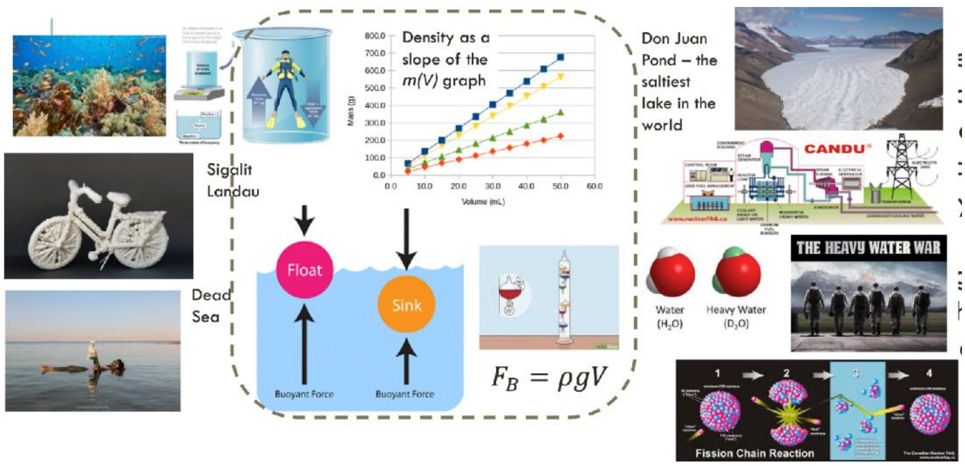


**Figure 3: Various applications of the concept of buoyancy in science, art, history, geography, and engineering**

For example, a number of variously shaped wooden blocks that have different volumes and masses can be used and their densities calculated. The students can also find objects of similar volumes but various densities. Another possibility is to use a PhET simulation application (Fig. 4) called Density (PhET Research Team, 2023).

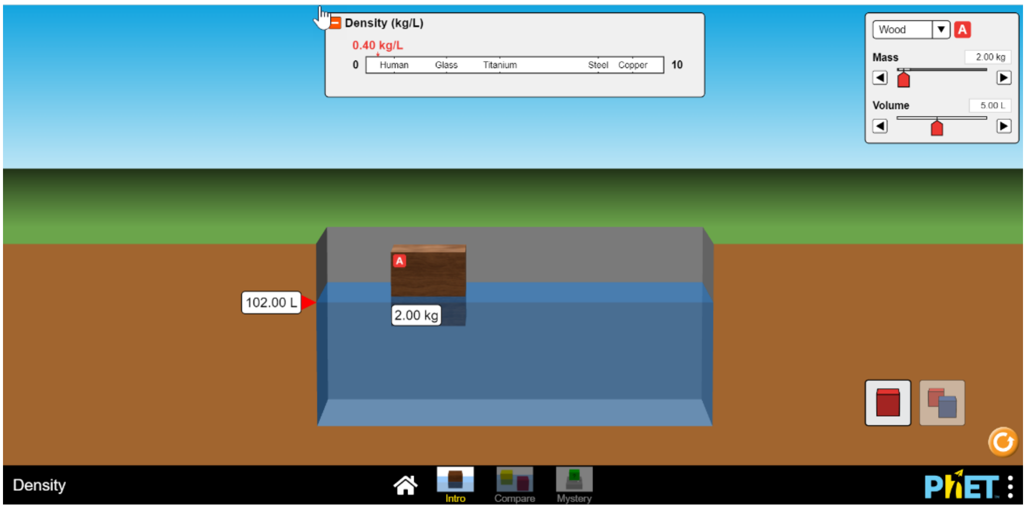


**Figure 4: A screenshot of PhET computer simulation that allows students to examine the concept of density and model the relationship between an object's volume, mass, and density**

The teacher organizes students in groups, recognizing their strengths, interests, and experiences. Some group members may be asked to approach the problem as a physicist, the other as a chemist, a biologist, and a mathematician. They need to work as a team in accordance with Gardiner's (2020) ideas. The teacher instructs students on the meaning of epistemic control and coordinates the whole class discussion.

<u>Stage III:</u> During this stage, students begin their work in groups to develop a plan of how they will investigate buoyancy, what information they might require and how they will represent it symbolically, graphically, and verbally. The teacher might support them in sharing relevant resources or intervening when necessary.

<u>Stage IV:</u> The groups propose different working hypotheses that describe the force of buoyancy. The students also discuss how they can model this force and how they can conduct experiments to test their models. They propose the experiment, including the ways of data collection and analysis. Thus, they attempt to describe the phenomenon of buoyancy quantitatively and justify how certain physical quantities might influence it. The teacher supports students, encourages them to compare and contrast their views, but does not provide the answers. For example, all "physicists" in the class may critique the proposed hypotheses.

<u>Stage V:</u> During this stage, the students test their hypotheses and modify them accordingly. They also consider various applications of buoyancy. For example, how it might explain the rise in the water level as a result of melting polar icecaps or how it can be used to understand different weather patterns or used in visual art or cooking (Fig. 3).

<u>Stage VI:</u> In this stage, the students consolidate their knowledge and might decide to return to the previous stages. The teacher helps them not only to summarize what they have learned but also uncover unanswered questions or potential misunderstandings. For example, the teacher might point their attention to the role of the force of buoyancy and the fact that the density of ice is smaller than the density of water in the marine life around us. They might ask new questions or decide to explore different applications of this phenomenon in various STEM subjects or in everyday life. The students might also collaborate on sharing their knowledge with others through poster presentations, group jigsaw discussion or other approaches.

## Example 2: Investigating motion through modelling

Details of this modelling activity are available in the facilitator's guide for modelling in STEM (Milner-Bolotin & Martinovic, 2022), a document we created to support professional development of teachers. To describe motion, one needs to know the object's position at all times. To do so, one must account for how fast and where the object is moving, as well as its initial location and time when the motion started. For example, a car can move with the speed of 10 m/s northward or southward. While the car's speed is the same in both cases, its direction is not. To describe this motion, we need to account for the direction, as well as the speed, which in physics is described by the concept of velocity. Why is that so? If you drive for 10 minutes northward and then for 10 minutes southward, you will end at the same place you started; at that moment your car's displacement (change in position) will be 0, although you spent 20 minutes driving and travelled the total distance of 12 km. An important skill, even for middle school students, is to be able to distinguish between average vs. instantaneous speed, speed vs. velocity, reading a map vs. reading a graph, and comparing ratios.

Throughout the activity, the teacher pays attention to the common student difficulties (Doorman, 2005): the difference in which the term "average" is calculated in physics compared to mathematics; that for curves and lines,

> The average rate of change over an interval is calculated in the same way for both graphs, but the instantaneous rate of change is not. For curves you have to take a limit, draw a tangent or derive a function, while for straight lines all instantaneous rates of change are identical to the average rate of change. (p. 24)

The fact that standing still is not represented as a point but rather as a horizontal segment on a position/time and speed/time graphs (in the latter case it coincides with a segment on *x*-axis) is also confusing. Doorman writes, "Standing still and moving with a constant velocity are similar from a graphical point of view. Both graphs have a horizontal velocity-time graph" (p. 25). Motion detectors could help students to embody graphs as representations of their movement. Video recording motion and analyzing it collectively would help clarify or alleviate possible students' misunderstandings. More details of this activity and how it can be organized in accordance with the EF4M in STEM can be found in English and French at the Mathematics Knowledge Network website (Milner-Bolotin & Martinovic, 2022).

## Conclusions

The education research community is still trying to unpack STEM teaching at various levels of schooling. In our recent book chapter (Martinovic & Milner-Bolotin, 2022), we describe five different approaches to organizing STE…M (e.g., STEAM, STEMM) programs and learning environments, including their feasibility for already existing teacher education and professional development programs. Our multi-year collaboration has shown that a way forward in any authentic multi-disciplinary teaching approach may be to "re-emphasize the nature...of [each] STEM [discipline]—as a sense-making activity" (Li & Schoenfeld, 2019, p. 1) and to strive towards enriching the students' experiences of the discipline.

Modelling activities enhance student engagement in learning and provide opportunities to integrate STEM subjects meaningfully and productively (Martinovic & Milner-Bolotin, 2021). Once the students create their models, these need to be tried and tested in a real environment. In other words, students conduct simulations, succinctly described in Alan Pritsker's (1989) keynote conference address:

> Simulation works because we abstract reality and because we, as problem solvers, are realistic modelers and analysts. We combine the engineering concepts of design and control with the experimental approach of the scientist. We use mathematics to solve problems and verify solutions. We see problems as opportunities. We are not hung up with optimization because we know our models are approximate. We don't worry about pitfalls until we are in sight of the solution. …We build models, use them, make a recommendation based on simulation results and implement the recommendation including the measurement of improvements obtained. We support on-going and continual improvement not just improvement (p. 2).

In Figure 3, we showed some ideas for making cross-disciplinary connections when teaching buoyancy. However, when engaging students in learning concepts that could be approached from different angles, it is important to consider disciplines' epistemological underpinnings and the perceptions of all the stakeholders (e.g., teachers, teacher educators). It is not only about learning the content; it is also about changing perceptions! Teachers need to experience STEM education as learners first and would benefit from having a framework that ties STEM pedagogy, epistemology, and curriculum together. Our EF4M in STEM can fulfil this role as it applies to teaching either separated or integrated STEM disciplines and opens opportunities for teacher collaboration. The inquiry courses that are part of many contemporary teacher education programs are well suited for implementing this framework. For example, the Teacher Education Program at the University of British Columbia has three 3-credit inquiry courses that could be used for promoting interdisciplinary STEM education (https://teach.educ.ubc.ca/).

One approach to dealing with the stated issues is to take part in a STEM Discipline-based Education Research (DBER; Henderson et al., 2017) movement. It is grounded in the idea that education in each of the STEM disciplines benefits from research that unites the specific content, culture, and methods of the discipline with the general discipline of education research. The authors further envision establishing a cross-discipline STEM DBER alliance, as a way for improving STEM research and teaching, and for creating a unified voice to dialogue with policy makers.

In order for an activity to be authentically STEM, more than one subject in it needs to become visible (Niss, 1994) and not just be used unconsciously as a utilitarian tool. For example, using a quadratic equation formula to find the time it takes an object in a free fall to reach the ground, does not make this physics activity, STEM activity. However, when mathematics modelling produces new insights, generalizations, or deeper understandings of scientific concepts and their applications (Williams et al., 2016), then it becomes STEM learning. In this case, a relatively simple mathematical derivation shows why uniformly accelerated one-dimensional motion produces a parabolic dependance of position on time, and a linear dependance of velocity on time. We call on educators to consider how modelling can open new authentic opportunities for meaningful learning making all STEM subjects and their interconnections not only important, but visible.

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# Bridging the practice-to-research chasm for in-service STEM teachers

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*Education reforms encourage mathematics and science teachers to leverage student thinking through inquiry practices. These practices focus on developing students' understanding of mathematical and scientific concepts to develop their disciplinary practices and fluency in core ideas, how this knowledge is constructed and the socio-scientific implications of STEM processes. However, teachers need meaningful professional development to develop inquiry teaching skills. In this study, I present an analysis of an academic year-long collaborative learning community. Results indicated that when novice and experienced teachers co-construct inquiry practices, novices assume the role of expert as they model inquiry for experienced teachers who are yet inquiry novices. This dynamic intermingling of roles can generate pedagogical innovation. This paper extends research on how collaborative professional development can generate supportive structures for STEM high school teachers, and help scholars and policymakers understand complex professional learning dynamics.*

*Keywords: Professional learning community, STEM teachers, inquiry-based teaching.*

## Introduction

Educational reform curricula encourage STEM teachers to incorporate critical thinking strategies into their lessons while introducing elements of science, technology, engineering, and mathematics (STEM). In Quebec, Canada, this includes proposing solutions to real-world problems, research, open-ended experiments, and engagement with interdisciplinary projects (Ministère de l'Éducation du Loisir et du Sport, 2007). Furthermore, UNESCO urges teachers to incorporate these pedagogies into their professional repertoires to facilitate students' ability to translate this knowledge into action (United Nations Educational Scientific and Cultural Organization, 2021).

The focus of inquiry-based teaching practices is to support student thinking by emphasizing practices such as posing questions, designing investigations, analyzing data, and developing evidence-based models from data (Windschitl et al., 2018). Inquiry-based (IB) teaching can leverage student thinking about mathematical and scientific concepts to develop their disciplinary practices and fluency in core ideas while helping students embody positive identities as STEM learners (Lampert et al., 2013; Windschitl & Stroupe, 2017).

IB teaching practices in mathematics classrooms take time to assimilate and perfect (Jao & McDougall, 2016), and in science classrooms, they "are rare, even in the classrooms of experienced teachers…[where] there is a focus on activity rather than sense making and that questioning, in general, is among [the] weakest elements of instruction." (Windschitl et al., 2012, p. 881). Instead of the dialogic co-construction of learning that is the foundation of inquiry-based learning (IBL), predominant classroom discourses follow the *Initiate-Response-Evaluate* model of questioning where teachers initiate a question, a student responds, and the teacher evaluates the response (Nystrand et al., 2003; Sherry, 2018; Windschitl et al., 2018). In contrast, implementing IBL asks that teachers develop professional practices that are grounded in educational theory, so they can initiate classwork that incorporates inquiry process skills, collaborative work, analysis, and synthesis of STEM concepts.

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Regrettably, there has been a dearth of professional support for in-service teachers to develop these sophisticated teaching skills, and thus their enactment remains problematic (McPherson, 2022; Windschitl et al., 2018). High-quality collaborative professional development (PD) such as a professional learning community (PLC) can facilitate expanding teachers' epistemological and pedagogical knowledge (Darling-Hammond et al., 2017; Hargreaves, 2019; Kruse & Johnson, 2017; Schmidt & Fulton, 2015), which is essential if teachers are to develop these practices.

This paper examines how in-service science and mathematics teachers collaborate to co-construct and enact inquiry-based pedagogies. This study explores two research questions. The first is "what were the tensions that teachers experienced as they struggled to develop inquiry teaching practices?" The second question is, "how did teachers' epistemological beliefs about teaching shift as they engaged, shared, and critically examined their professional practice in a PLC?" Understanding how teachers work in a collaborative PD model can help scholars and policymakers understand professional learning dynamics. This work extends current research on how STEM high school teachers support each other in a PLC.

## Methodology and methods

This study employed ethnographic methodologies to analyze the world of teacher participants, the social and cultural structures, and processes that they engaged with as they developed inquiry-based teaching strategies (Davies, 2008; Van Maanan, 2011). Participant discourses highlighted their school cultures and how a community of teachers might reshape this culture. Through an ethnographic approach, I explored participants' fears, doubts, and frustrations with their professional practice to understand how their tensions of practice could be redressed as they worked to shift school culture.

I followed purposive sampling methods (Maxwell, 2013; Seidman, 2013) to recruit eight high school mathematics and science teachers. Teachers participated in an eight-month PLC. from four Anglophone Québec high schools. Participants included three novice and five experienced teachers. The novice teachers were in their first or second year of teaching, while the in-service teachers' range of experience was from 15–26 years. Some teachers taught science while others taught most of their teaching load in mathematics with one science course added. Ayesha, a first-year teacher was trained to teach science and accepted an art teaching position halfway through the school year (see table 1).

**Table 1: Profile of PLC participants**

| Teacher | Nancy | Liam | Vera | Giulia | Carol | Steven | Zofia | Aysha |
|---|---|---|---|---|---|---|---|---|
| Years Experience | 15 | 26 | 15 | 1 | 1 | 2 years part-time | 0 | 0 |
| Teaching Subject | Grade 10 Science | Grade 9 Science | Grade 10 & 11 Science | Grade 9 Math, Grade 10 Science | Grade 10 Science | Grade 7 & 9 Math, Grade 10 Science | Grade 8, 10, 11 Science, Grade 8 Math | Substitute Science and Math Teacher Grade 7-9 Art |

Participants from four schools were part of the public sector; one was private. Two of the four public schools were large, exceeding 1,500 students. The two public schools and the private school were smaller, with populations of fewer than 300 students. All schools were in bilingual suburban communities. Teachers' motivations for joining the PLC included an interest in developing their reform practices, and they were enthusiastic about learning through collaboration with colleagues.

Eight PLC meetings of 2.5 hours occurred on average once per month from October to May in the afternoon. Teachers' focus was on developing inquiry-based teaching practices. During meetings, teachers discussed inquiry-based teaching practices following a video club format in which teachers discussed and analyzed how they enacted inquiry-based pedagogies (Borko et al., 2008; Horn & Little, 2010; van Es et al., 2014). Eight of the teachers shared two 10-minute segments of video-recorded classroom practices followed by participant discussions and analysis of observed inquiry practices. Aysha was not able to video-record teaching lessons as she was not teaching full time, and Liam was unable to generate a second teaching video because of time constraints imposed by end-of-year high stakes exams.

In the PLC meetings, teachers discussed questioning techniques, focusing on eliciting, pressing, and probing student thinking, generating opportunities for students to co-construct evidence-based explanations based on observations and data, and positioning students to compare scientific explanations to the accepted scientific phenomenon.

The data corpus included 16 semi-structured pre- and post-PLC interviews that were video-recorded and transcribed. The purpose of the initial interview was to gain a general sense of teachers' professional backgrounds, and epistemological beliefs and pedagogical practices. The second individual interviews focused on a PLC's effectiveness in developing inquiry-based (IB) practices, and on how participants negotiated the work of collaborative learning in a PLC. PLC activities, interactions, and informal conversations were video recorded, and portions of the meetings that focused on teachers' collaborative discourse were transcribed. The portions of PLC meetings where teachers presented their video recorded classroom teaching activities were not transcribed.

I developed salient concepts into themes using the constant comparative method (Braun & Clarke, 2006) and I analyzed and interpreted the themes using an inductive thematic approach to theorize patterns relating to the research questions (Braun & Clarke, 2006).

## Findings

Two central themes emerged from the data: teachers' struggles to enact inquiry-based pedagogies and the importance of supportive structures during an *in-situ* learning experience. The professional trajectories of the PLC participants fell into two very different groups. The first group included three experienced teachers. The second group, the five novice teachers, had been taught IB practices in their pre-service university program.

## Experienced teachers

The first interviews with experienced teachers suggested they were neophytes when it came to inquiry-based practices, which was not surprising, given the dearth of PD opportunities that focus on developing professional practices *in situ*.

The pre-PLC interviews with the three experienced science teachers captured their epistemological stance to IB teaching. Of the experienced teachers, Nancy alone expressed an interest in incorporating IBL into her teaching repertoire, and she was the most open and eager of the experienced teachers to update her practice:

Nancy: I haven't done it as much as I would like. I need to practice it more.

Vera, a confident and successful teacher, was openly dismissive about the potential to use IBL:

Vera: A time constraint. I feel that that's just a waste of time.

In the initial interview, Vera talked about her high success rate on end-of-year exams, which, in her opinion, validated her reliance on traditional teaching practices based on lecturing and note-giving.

Liam, like Nancy and Vera, was aware of IBL. He was on the fence, believing that his practice of talking with his students and asking multiple IRE questions was sufficient. He had rejected incorporating IBL into his lessons because he lacked the training, although he admitted that "Teaching the same old way that we've always taught before is not necessarily a good thing."

The experienced teachers made significant strides forward with questioning skills—pressing students, eliciting student thinking, and using a driving question to anchor lessons. For example, during a PLC meeting, Nancy discussed how she developed IBL in a grade 10 science class. Her students engaged more frequently with cognitively demanding peer discussions:

| | |
|---|---|
| Nancy: | They're starting to think more. I need to practice. It's gonna take a couple of years. Once I'm comfortable, it'll be better. |

In this excerpt, Nancy suggested that it takes time and practice to develop IBL, which is consistent with the literature—mastering IBL is time-consuming, and requires ongoing PD, which is one reason that these practices are rarely seen, even in the classrooms of experienced teachers (Windschitl et al., 2018). Nancy also noted that introducing a novel pedagogy such as IBL poses difficulties to students as well as the teacher since students are unused to being questioned, pressed, and probed for high-level thinking.

Vera, like most of the PLC participants, chose to work on eliciting student thinking by pressing and probing students for understanding while maintaining a public record of their thinking. Vera, who had identified as a "confident individual" in the first interview, was less than confident when she attempted to teach a physics investigation of lenses using IBL. During a PLC meeting, she articulated her discomfort with incorporating IBL into her lessons.

| | |
|---|---|
| Vera: | So yeah, it's not my finest moment, but clearly, I conveyed the issue the message because they all did well on their test [laughter]. |
| Author: | So, you said it's not your finest moment. What did you think wasn't fine about it? |
| Vera: | I didn't feel as prepared and organized...Like I was just like, okay, I gotta make sure I have to mention this. Ask more questions like I was trying to think of all the things that you are telling us as opposed to just being in my own head. I had other heads talking to me [laughter]. So, at the same time, it's like, oh crap, I forgot what slide I have next. And then, now oh no, I have to remember to do this, and so it just I just got lost up here. And as a result, the that's why I was like, oh, oh, wait first. Talk about this. So, I just got lost. |
| Author: | So, what would you like to improve on as you watched that? |
| Vera: | The voices in my head [laughter]. |
| Author: | But they should always be there, right? |
| Vera: | Yeah, but it would be nice if it was just mine. Like I don't mind if the multiple personalities of me are talking, but not the other ones like, oh my God, I can't disappoint Heather. And I'm not writing things down. They always have good questions, but I was like, aghh. And so, I just stopped. Putting myself in one big endless loop. So, if I can just tune all that stuff out, then I have no problem. |

It is clear from Vera's self-analysis from the PLC transcript that she was uncomfortable with a new style of teaching. As she got lost in the process of IB teaching, she said that her thinking was disorganized. She struggled to sustain the logic of her lesson, which she lost as she focused on IBL. However, she remarked on her evolving practice, celebrating her progress:

| | |
|---|---|
| Vera: | Now, I do a progression of teaching, as opposed to shoving everything at the beginning. |

Overall, the experienced teachers reported changes in their pedagogy. Nancy's process with IBL was methodical, incorporating new teaching techniques in incremental steps as she acknowledged that the process of developing her professional practice would likely take a few years. Additionally, Liam spoke about changes to his teaching, although he also acknowledged that not much had changed. He decided not to incorporate driving questions in his daily practice, and he did not experiment with model-based inquiry. However, he was more mindful of elevating his questioning techniques, and committed to further developing his IB practices.

As evidenced in their teaching videos, the experienced teachers made significant strides forward with their inquiry questioning skills—pressing students, eliciting student thinking, using a driving question to anchor lessons.

## Novice teachers

How did the five novice teachers fare with IBL? In the first interview, Guilia, who teaches grade 7 math and grade 10 science, described her teaching approach:

Guilia: I'll present the material first either through PowerPoints or through traditional blackboard whiteboard teaching, and then we'll do practice questions together, and then they'll try some on their own…and then after that we'll have a review, more questions and so forth, and then a quiz, test.

Guilia spoke openly about her traditional teaching experiences in the first interview. Carole's experience with IB teaching was like Guilia's. Carole believed:

Carole: I like giving a little part of a lecture, notes so I would do a little bit of that, but I truly believe and after doing a lot of hands-on or practice so I would try to always find an activity that goes directly with the lesson. Sometimes my activities—I call them activities but, in the end, they ended up being just like handouts, you know it was like extra practice, which eventually doesn't, I find, was not like hands-on relevant.

The novice teachers lacked the confidence to use their university training and were relying on traditional modes of lesson delivery—lecturing, pen-and-paper, fill-in-the blank handouts interspersed with infrequent low leverage lab situations in science classrooms, and mathematics lessons centred on identifying problems of understanding, then fixing, and solving students' misconceptions. Of the eight participant PLC teachers, Aysha had the clearest vision of IBL. However, referring to both of her field experiences (FE) she noted the following:

Aysha: [I] didn't do much inquiry-based, it was more just lecture and lab. I would say like 50% lab, 50% lecture, so obviously I think I still haven't had the chance to do what I've been wanting to do.

Aysha, like Guilia and Carole, did not have an opportunity to practice IBL during their FE. After talking with the novice teachers, my impressions were that they were drowning. Aysha, Carole and Guilia had practiced IBL at university during their method courses. However, as Carole pointed out, these practice rehearsal sessions were artificial as they unfolded in front of other university peers, which was not an accurate representation of a high school classroom teaching experience. Ideally, the novice teachers would have had multiple opportunities to practice IBL during their FE. However, their cooperating teachers (or associate teachers, as they are called in Ontario) had not been trained to use IBL, and thus, their cooperating teachers were traditional in their classroom practices. The lack of time given to experimentation with IBL left the novice teachers with few IB experiences to draw upon as they graduated and began their teaching careers.

Zofia and Steven were teaching math and science courses. In the first interview, Zofia discussed her approach to teaching:

Zofia: A lot of the times like forty lecture and twenty discussions.

Like Zofia, Steven's approach to teaching mathematics at the beginning of the year was:

Steven: A teacher center [sic] approach. Per class, usually 20 to 30 minutes of giving notes and lecturing.

The remainder of his class time was spent allowing students to complete pen-and-paper practice sheets. Neither Zofia nor Steven discussed questioning techniques or engaging mathematics students in discourses centred on mathematical concepts.

At the end of the PLC year, Guilia, Carole, and Zofia were flourishing. Guilia's second interview recorded a significant shift in practice. She described the job of the teacher as setting up learning situations in which students had opportunities to discuss their intuitive ideas regarding scientific concepts, that her focus now was on the elicitation of these intuitive and developing scientific concepts as she questioned students in a dialogic sequence of questions and answers. In her words:

Guilia: Students understand how or why something is happening. Not just by being told from the teacher why it's happening, to discover it themselves by tackling their own ideas and their preconceptions and misconceptions about something and that it's really unravelling that and then putting it back together in a way that truly explains how and why something is. Getting the students to tell me what they're thinking and then breaking that down.

In the second interview with Carole, she spoke of her earlier fears regarding IBL. However, she had moved beyond her feelings of trepidation, and was now confidently incorporating IBL into her teaching.

Carole: It was scary the way it was in my head, but through the PLC, it was actually like OK this is actually not that bad…I wanted to do it and I had heard about it but I never thought I was doing it, and I didn't want to do it. Now it's like actually, I can do this. I liked it. I saw the students in the time we had have discussions. Sometimes I feel like at the beginning I was going to lose them because I don't know where we're going. I don't know where it ends up, so scary. But then it's actually good, and it's was oh OK now I understand this.

Carole's initial discomfort with IBL is common. During IBL, the teacher is giving up control of the lesson to the students. In the literature, lack of content knowledge is one reason why teachers resist IBL (Feger & Arruda, 2008), which was a factor in Carole's earlier career decision to play it safe. The novice teachers, particularly Carole and Guilia, spoke of using student discourse to develop understandings of mathematics and scientific concepts. This was particularly evident in their teaching videos. Zofia's reflection about her second mathematics teaching video captured her developing understanding of IBL:

Zofia: I'm not going to say the answer, I wait for them. Whenever somebody says a wrong answer, I put it down, and let them figure it out...I acted surprised when I just thought, okay, we're going to put it down for them to see that it doesn't actually work.

The data suggested that the PLC experience broadened teachers' conceptualization of IBL as all of the participants began to use high-quality instructional practices. There was strong evidence that teachers were most aware of and incorporating the elicitation of students' intuitive ideas by pressing and revoicing student thinking. They were making progress with their ability to orienting students to each other's ideas and position students competently. Moreover, they were encouraging dialogue among students and supporting them to build their concepts/models. Furthermore, Carole, Liam, Nancy, and Vera enacted other aspects of IBL, such as engaging students in investigations, making observations, collecting data, constructing data-based explanations, and comparing students' explanations to scientific models. Overall, teachers demonstrated a change in their approach to their professional practice as they reconstructed their professional practice through their enactment of IBL. Collectively, the teachers in the PLC were able to shed aspects of their pre-reform practices.

Last, the findings illuminate how the PLC helped experienced teachers visualize inquiry-based pedagogies. Vera referenced the novice teachers' videos, who, she thought, demonstrated greater comfort with this style of teaching, as seen in the following PLC excerpt:

Vera: By watching the videos [of novice teachers] and I became more conscious.

Additionally, the novice teachers overcame their insecurities and developed confidence through encouragement, validation, and support from their experienced colleagues. In Guilia's words:

Guilia: The PLC acted like a motivational point for me to continue doing this because you get to go and meet these other teachers who are trying to do the same thing and talk about it. So, if you see all these other people doing it, you kind of want to join in as well. So, it is motivational.

Together, the community or practitioners developed an understanding of IB teaching. They were able to leverage their different experiences of professional practice to develop and innovate.

## Discussion and conclusions

This study follows the professional trajectory of five novices and three experienced science and mathematics teachers. What was particularly significant was how the categories of experienced and novice were frequently blurred, since learning occurred in two directions—from experienced to novice, and from novice to experienced. This PD experience initiated a shift in professional practices as the eight science and mathematics teachers began transitioning to inquiry-based practitioners.

The learning during the PD suggests that all learners co-developed inquiry-based practices in the professional community. The novice teachers initially spoke at length about their struggles with inquiry-based lessons. They spoke of fear, worried that incorporating inquiry into their lessons could lead to a loss of class control, lack of time to deliver the curriculum when final exams are the end goal or following teaching practices contrary to their department norms. However, these narratives began to change as the novice teachers, guided by the experienced teachers, aligned their professional practices with their pre-service models of instruction. In sum, novice teachers gained confidence as they continued their journey to becoming inquiry professionals, benefitting from working with experienced teachers during the critical induction years. The experienced colleagues were able to mitigate the novice teachers' insecurities, anxieties, classroom management problems and lesson planning issues. At the same time, the novice teachers exemplified inquiry practices because they had university experience with these pedagogies. The experienced teachers, who were neophytes to inquiry teaching, developed a burgeoning understanding of inquiry-based teaching by watching and discussing the novice teachers' video lessons. In essence, there was a dynamic weaving and intermingling of roles within the learning community. There were multiple instances where the novice and experienced teachers supported each other to co-develop these pedagogies. As seen from their respective learning trajectories, both the novice and experienced teachers developed new ways of teaching.

The focus of the PD was to help in-service mathematics and science teachers navigate the space between traditional teaching practices and inquiry-based pedagogies. The experienced teachers did not have opportunities to learn and implement inquiry-based pedagogies because there are few PD opportunities where teachers can learn, develop, and practice these pedagogies. High-quality PD should facilitate collegial discourse among teachers to support critical reflection and examination of professional practices (Hargreaves, 2019; Kruse & Johnson, 2017; Putnam & Borko, 2000). Effective teacher learning models emphasize that teachers learn best when they have opportunities "of creating, developing, organizing, implementing, and sharing their own ideas for school change rather than being passive recipients of knowledge from the outside" (Lieberman et al., 2017, p. 1). However, PD activities that focus on developing teacher practice require a substantial amount of time, planning, and financial resources (Deneroff, 2016).

PLCs can provide a space for teachers to experiment with and develop practices that are immediately transferable to the classroom, in part because the longer duration of a PLC is more effective than episodic workshops. Thus, the PLC structure provides sufficient time to include applications to classroom practices, time for collegial discourse, coaching, modelling, observation of teaching, and reflection and feedback from colleagues through teacher learning situations that are flexible, teacher-centred, and interactive (Vescio et al., 2008; Webster-Wright, 2009).

This study extends the current literature on PD, suggesting that a PD model that intentionally includes novice and experienced teachers can help teachers co-create new ways of teaching, potentially leading to pedagogical innovation. This PLC experience initiated a shift in professional practices as teachers began transitioning to inquiry-based practitioners. The novice teachers' practices had lapsed to traditional teaching, informed less by their university coursework and more by the *apprenticeship of observation* (Lortie, 2002)—their professional vision informed by years spent learning in classrooms. In this study, the novice teachers had an opportunity to share their struggles with IBL enactment, their worries that they might lose class control, their experience a reduction in instructional time, and their perceived focus on following institutional norms. These narratives shifted as the experienced colleagues mitigated their insecurities, classroom management problems, and lesson planning issues. Simultaneously, experienced teachers developed an understanding of IB teaching by watching and discussing the novice teachers' video lessons. There was a dynamic weaving and intermingling of roles within the learning community.

The data raises relevant concerns. First, educators need to understand the disconnect between university theory and the reality of on-the-job teaching. The novice teachers could not enact the theory of inquiry-based practices without support. Instead, they had lapsed to traditional teaching, informed by the *apprenticeship of observation* (Lortie, 2002)—the years spent observing their classroom teachers.

## Limitations

This study examined the professional trajectory of eight teachers of mathematics and science as they engaged with and worked to reposition their professional practices in the same geographic area. Thus, the findings provide a local or regional snapshot of professional development and learning. It is quite probable that had the research been conducted in another region of the province, that the findings would have been different, depending on the context, group dynamics, and teachers' goals for the PLC. Furthermore, in Québec, the domain of PD falls under the jurisdiction of each school board, and therefore the findings from this study may not apply to other school boards in Québec because each board enacts a unique PD vision.

Additionally, this study did not examine the transference of teachers' pedagogical gains to student learning in the classroom. The focus of this study was to examine how STEM teachers develop in a professional learning community. Logically, the final objective is to improve student learning and engagement with mathematics and science. Future studies need to examine the impact that reformed teacher praxis has on student achievement. Although educational scholarship has written much about PLCs and their potential to develop teachers' professional practice, more needs to be done.

## Implications

The focus of this study was to help STEM teachers develop inquiry practices. This study extends the current literature on PLCs, suggesting that a PLC model of PD that includes novice and experienced teachers is particularly useful since novice and experienced teachers have different experiences with professional practices. Including teachers of different experience levels in a PLC generates dynamic supportive structures that facilitate teachers' potential to navigate the space between traditional and reform-based pedagogies. This research can inform researchers, educators, school boards, and policymakers how a heterogeneous group of mathematics and science teachers engage, learn, and enact useful and effective processes that facilitate professional growth.

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# Scaling up Kanga-Kids training program for math teachers as part of a city STEM ecosystem: First trial

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*This paper presents the first trial for scaling up the Kanga-Kids Training for Math Teachers in early grades as part of a municipal STEM Ecosystem. Following a brief overview of the Kanga-Kids Training Program for Math Teachers and its evaluation findings during its first and second years, we present an analysis of the scaling up process in five primary schools. The results and conclusions are presented through a case study methodology that showcases what works best and what needs to be improved in the scaling up process. The paper draws two main conclusions: first, theory and practice occasion iterative cycles of enactment, and second, the success of the program will be achieved if it becomes mainstream through policy support.*

*Keywords: Scaling up, STEM ecosystem, training program.*

## Introduction

Kanga-Kids is part of the first STEM ecosystem springboard and is based on professional development workshops for early childhood educators who then facilitate the program in their classes and are expected to share their knowledge and experience with their colleagues in schools.

All data presented here is based on the second-year program evaluation report requested by Be'er Sheva Municipality STEM Ecosystem which initiated and supported the project.

The STEM Ecosystem is a network linking stakeholders in and around the city in a collective effort to advance STEM education and the development of 21st-century skills. Today's children will need a different set of skills than those needed in the past to enable them to thrive in the changing world of work possibilities. These 21st-century skills include, among others, science and technology literacy as well as work habits and qualities such as collaboration and teamwork, creativity and initiative, adaptability.

## Literature review

STEM offers an interdisciplinary approach to learning where content is coupled with real-world lessons as students apply science, technology, engineering, and mathematics in a context that makes connections between various aspects of their lives (Lantz, 2009). The quality of children's learning environments influences later academic success (Campbell et al., 2001; Hadzigeorgiou, 2002). Thus, appropriate STEM experiences in early childhood can be starting points for supporting children's continued success in STEM at elementary, secondary, and postsecondary levels. Furthermore, the National Science Teachers Association (2014) suggests that early childhood education may offer

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opportunities for teachers to engage in science and engineering activities that capitalize on students' interests, experiences, and prior knowledge in natural extensions of purposeful play.

Various goals of STEM education for student development have been proposed, including students' knowledge of the fundamental concepts relevant to STEM, their understanding of the characteristic features of STEM disciplines, their acquisition of skills addressing STEM-related questions and problems, capabilities relevant to the 21st century (such as creativity, critical thinking, communication, and collaboration), and positive attitudes (such as interest, engagement, and self-efficacy) toward STEM (Bybee, 2013; National Research Council, 2014). Until now, a great deal of research has been conducted on the positive impacts of STEM education at school levels, such as the effect on academic achievements of high school students and their career choices (Han, 2017; Han et al., 2015), and elementary and middle school students' dispositions toward STEM (Afriana et al., 2016; Christensen et al., 2015; Guzey et al., 2016).

The necessity of early exposure to STEM was highlighted by several scholars (Bagiati et al., 2010; Bybee & Fuchs, 2006). It was argued that young children are congenitally curious, creative, and collaborative, which are the same attributes needed for STEM education, and these attributes in young children make them naturally interested in STEM-related concepts (Banko et al., 2013; Reis & Renzulli, 2009). Furthermore, children have innate intellectual tendencies that enable them to learn STEM, such as the ability to make sense of experience, analyze, hypothesize, and predict (Katz, 2010).

Early childhood science instruction should build on children's curiosity and ability to make sense of experience. It should include an inquiry approach and provide appropriate platforms that promote conceptual understanding and reasoning (Leuchter et al., 2014; Roth et al., 2013). Eshach and Fried (2005) argue that science is an important—and perhaps imperative—component of early childhood education because it builds upon students' innate interests in the natural world, can help develop positive attitudes towards the discipline, and can provide a foundation upon which further learning and understanding can be built.

However, research suggests that current professional development systems are largely ineffective and make little or no impact on teacher behavior or child outcomes (Farkas et al., 2003; Joyce & Showers, 2002; Snyder et al., 2011). While traditional methods of professional development such as training sessions, workshops, and conferences have been found to increase teachers' awareness, these forms of professional development are not associated with teachers' sustained use of research-based interventions (Artman-Meeker & Hemmeter, 2013; Odom, 2009). Alternative research-based professional development is critical. Assessments demonstrated that the provision of high-quality professional development has shown significant improvement in young children's achievement (Brendefur et al., 2013; Kermani & Aldemir, 2015). Professional development should be ongoing and appropriate to the subject matter being taught; it should include opportunities for teachers to actively participate and have some relevance to what is happening in the classroom (Garet et al., 2001).

However, many early childhood teachers are neither eager nor prepared to engage children in rich experiences in domains other than language literacy (Duschl et al., 2007, Clements & Sarama, 2014; Brenneman et al., 2009). In fact, widespread anxiety about topics such as mathematics exists among teachers of young children and correlates with their students' achievements (Beilock et al., 2010). Furthermore, many teachers do not know how to adapt STEM instruction to suit the needs of their students.

Many teachers continue to hold negative feelings about math and science even after graduation. In mathematics, for example, these feelings lead to undervaluing the teaching of math, avoiding or minimizing math instruction, and ineffective ways of teaching the subject (Huinker & Madison, 1997; Lee & Ginsburg, 2007). Similar trends appear in science education. Consequently, we need to effectively increase teachers' STEM knowledge and change negative dispositions and beliefs through high-quality pre- and in - service professional development.

## Be'er Sheva's STEM ecosystem

In 2017, several philanthropic funds led by the Jewish Funders Network (JFN) collaborated to design and establish a STEM ecosystem in Be'er Sheva from preschools until the student reaches the employment market.

The design process of the Ecosystem began in 2018 and was headed by the local authority in cooperation with TIES (Teaching Institute for Excellence in STEM)— an organization that develops and implements the model in the USA and other countries worldwide.

The leadership team is headed by the deputy mayor of Be'er Sheva, who holds the Education Portfolio in the Municipality. The vision of the municipal ecosystem is to create a platform of partnerships that will contribute to the optimal actualization of the vision of turning Be'er Sheva into a top-tier STEM metropolis that will produce a continuum of opportunities and outcomes from preschool age to successful future career paths. The purpose of the municipal ecosystem is to promote the human capital potential of residents of Be'er Sheva and its metropolis from early childhood through high school in STEM-related fields.

## The Kanga Kids program

The Kanga-Kids is a program based on professional development workshops for early childhood educators who then facilitate the program in their classes and are expected to spread their knowledge and experience through their colleagues in schools.

Materials in the Kanga-Kids program do not overlap with the schools' curricula, are oriented to the development of children's logical thinking, spatial abilities, and high-order thinking skills, and are based on puzzles that are connected to their everyday lives.

Two examples of the tasks used in the program are presented below:

*Four identical pieces of paper are placed as shown. Michael wants to punch a hole that goes through all four pieces. At which point should Michael punch the hole?*

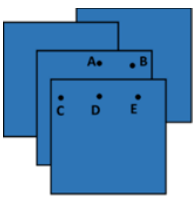


**Figure 1: Four pieces**

*A student made the shape shown using 12 cubes. He put one drop of glue between any two cubes that share a common face.*

*How many drops of glue did he use?*

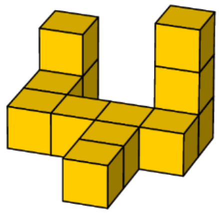

**Figure 2: 12 cubes**

The main goal of the program is to expand STEM learning among children, provide professional development for educators, and create communities of practice to share experiences and promote best practices.

Providing quality opportunities to explore STEM content outside of formal school settings removes the academic pressure and fear of failure that can contribute to STEM disengagement, even among bright and motivated students (Potvin & Hasni, 2014). It also supports positive youth development — including fostering quality relationships with peers and adults and acquiring other social skills — by offering a safe place for children to learn and play when their primary caregivers are at work or otherwise unavailable (Noam & Triggs, 2019).

Research literature shows growing evidence that participation in high-quality, STEM-focused programs can positively change youth attitudes related to STEM engagement, identity, career interest, and knowledge of careers (Allen et al., 2019; Chittum et al., 2017; Young et al., 2016).

During its first year, Kanga-Kids operated an ongoing collaborative training, through which it succeeded in changing participants' attitudes toward math and teaching it. Moreover, it succeeded in changing participants' teaching routines after the in-service training (for more details, see Appelbaum & Zamir, 2022). Analyzing and conceptualizing those findings led to the conclusion that training through modelling methods and fostering creativity are basic to reaching professional development. When talking about creativity, we mean fostering a process of exploration, play, risk-taking, mistake-making, self-evaluation, and feedback (Appelbaum & Zamir, 2022; Runco, 2014; Sternberg & Williams, 1996). All these principles were implemented in the Kanga-Kids in-service teachers training course. It was shown that "intentional actions…to create STEM learning environments, build STEM capabilities and nurture STEM dispositions" (Murphy et al., 2018, p. 4).

This paper focuses on the second year of work, in which the evaluation of the program investigated the scaling up activity and its impact: the scaling up was the first trial to disseminate the program (McLaughlin & Mitra, 2001). The basic assumption was that the knowledge and skills acquired by the first group during the training program could enable them to disseminate the program and its impact by starting a scale up phase.

Fifteen teachers chose to attend the Kanga-Kids in-service teachers training course. All participants were certified teachers, of whom eleven with Bachelor's degrees and four with Master's degrees; all were aged between 35 and 50 and were recommended by their principals who highly appreciated them as future leaders in their expertise. They all taught at elementary schools (1$^{st}$ to 6$^{th}$ grades). It is important to stress that each of their classrooms comprised 32-35 students in extremely heterogeneous socio-economic groups.

Eight four-hour workshops over the course of one semester were aimed at delivering modules that focused on mathematics concepts and different strategies of problem-solving (Appelbaum & Zamir, 2022).

Five outstanding teachers were chosen to introduce STEM skills to the staff in their schools in a way similar to that in which they acquired the same skills during their own professional development. They were enthusiastic during the training process and presented high-level outcomes of their work. Adam, the first-year coach, met them during the year both one-on-one and in a collaborative-learning group.

The scaling up phase became an interesting study matter. Therefore, we have chosen to describe this in length through a five teachers' case study implementing thick description (Geertz, 1990).

## Scaling up

The educational process is a crucial aspect of human development, providing individuals with the knowledge and skills necessary to navigate the world around them. Scaling up concepts in the educational process involves the needed steps for expanding and improving the quality and reach of educational programs (Scriven, 1991) to provide equitable access to education for all. Scaling up is a challenging process with numerous barriers that must be overcome. A good example is found in Healey and De Stefano (1997) claiming that "after nearly two decades of school reform effort, fewer than 5% of the schools in the USA have changed their core educational practices" (p. 2). Taylor et al. (2011) claim that the complexity of scaling up has been neglected and that the whole educational system suffers from the fact it has not been paid enough attention. Scaling up is seen as a problem independent of human resources and budgets. Bodilly (1998) presents her view on the complexity inherent to scaling up. She highlights three factors that while identified as most important for the success or failure of scaling up educational reform, they are not the focus of evaluation:

(a) Origin: The origin of the reform design relative to adopters (internally, developed by teachers in the school, or externally by a developer).

(b) Target: A targeted reform toward a specific population or curricular area, or implemented broadly across sites.

(c) Object: Reforms aimed at structural or instructional change.

In 2022, the European Commission completed a toolkit for scaling up projects (Barnett, 2022). They designed seven steps starting from building a shared understanding across government; identifying and framing the expected challenges; searching for innovations with potential; assessing the evidence for these innovations; guiding the choice of scale-up pathway; promoting continuous learning; and working toward mainstreaming. Like Bodily (1998), the writers stress the fact that projects cannot reach the scaling up stage if they do not become part of the mainstream recognized by the Government. Therefore, policymakers are still looking for an effective way for replicating models that work. They have found that when acting in networks among schools, the possibility of replicating good models increases as found by Farrell et al. (2012) that presented the example of charter schools embracing 25 schools engaged in a scaling up project. In this paper, we follow the example of Cooley (2016), a member of a team from MSI (Management Systems International).

Our scaling up trial follows their guide stating three main steps:

Step 1: Develop a scaling up plan,

Step 2: Establish the pre-conditions for scaling up,

Step 3: Implement the scaling up process.

## The scaling up program description

Steps 1 and 2: the scaling up program relied on the assumption that experienced teachers, who personally experienced the Kanga-Kids training program would be prepared to disseminate the program. In their conclusions, Appelbaum and Zamir (2022) note that, Kanga-Kids adopted and implemented Brenneman's ten best practices of professional development to support the program: (1) included educators and administrators in the ongoing design; (2) included professional supports; (3) boosted teachers' content knowledge; (4) took into consideration teachers' attitudes and beliefs; (5) engaged with teachers on different levels (large and small groups, one-to-one); (6) connected the material with relevant classroom practice; (7) involved educators in feedback and reflective coaching cycles; (8) established a collaborative learning community; (9) ensured that the program is ongoing and long term; (10) ensured that the material was individualized to suit the needs of the particular classroom (Brenneman et al., 2019). Like Bodilly (1998) and Farrell et al. (2012), Adam saw the ecosystem in parallel to a city network that would increase the possibility of replicating good models.

Step 3: during the scaling-up phase, the Kanga-Kids professional development model accomplished Brenneman's ten best practices through two main components: (1) workshops providing space for the new leaders to share and solve mutual professional problems and receive feedback (2) reflection circles providing individualized coaching, feedback, and support.

## Research questions

1. What was the vision behind the scaling up program design?

2. To what extent did the program graduates succeed in attaining collaboration in their schools?

3. To what extent did the program graduates feel competent for their undertaking?

4. How, and to what extent, has the program succeeded in changing teachers' attitudes and routines in the math classroom?

## Methodology

The scaling up phase is presented through the evidence brought by five teachers' case study. The case study was built within the qualitative paradigm to enable the participants to enunciate their own opinions and interpretations of the process. The case study is a methodology near the anthropologist or the clinician's work, and it can cover causation where statistics cannot give a deep explanation (Scriven, 1991). It is based on interviews with the graduates and the coach. All the interviews were video recorded with the participants' consent after verifying their comprehension of the research goals. The data was first analyzed based on grounded theory (Glaser & Strauss, 2008). The findings were then compared to other theoretical models such as Learning from Success (Rosenfeld et al., 2006) and, of course, Scaling up Phases in Change as in Cooley (2016), Bodilly (1998), and McLaughlin and Mitra (2001)

## Findings

We present here a case study about five teachers' stories through which we analyze the first scale up trial results. Each teacher's story is part of the whole case study puzzle enabling the researchers to reach new insights.

Five teachers with degrees in mathematics who had participated in the Kanga-Kids training program were asked to participate in the second phase and were directed to disseminate their knowledge to the other math teachers in their schools. Adam, the coach stated:

> After the training phase, I was very impressed by some of the participants' collaboration, participation, and initiatives. They also presented very interesting and complex outcomes from their work with their pupils in the classroom. There were seven teachers out of the fifteen; I wondered whether they would be ready for the next step. I talked with each of them personally. Two later left the area to work in new locations. I focused on the other five.

Adam's goal was to bring the idea of developing STEM skills in the first grades of five elementary schools by teaching mathematics via puzzles. His vision was to subsequently implement the STEM program throughout the whole municipal ecosystem. The five schools were seen as a pilot within the city ecosystem.

Similar to Cooley (2016), Adam suggested testing scaling up considerations as early as possible in the innovation process.

By the end of the academic year, it was clear to Adam and the participants that only two of the five had succeeded in promoting the process in their schools. The findings will be presented as one case study of two groups:

The two teachers who had successfully integrated the change in their schools.

Three other teachers who had not succeeded in leading their teams to a change but continued teaching toward STEM skills development using Kanga-Kids' materials and other puzzles in the mathematics classroom,

## Mary's story

Mary: Adam phoned me at the end of August, close to the beginning of the new school year...he opened a [WhatsApp] group and wrote personally to each of us, welcoming us to the group: 'I have chosen you [to participate] in the scaling up process for developing thinking skills in math." During the interview, Mary said that Adam held one-on-one meetings with each of them explaining the project and recruiting them for the task. Mary introduced herself as the head of mathematics and coordinator of the math team in her school. She has been coordinating and supervising the team for a long time and was very proud that Adam chose her to be part of the second phase and to help train the second teachers' group (together with a co-trainer).

Mary related that she had specialized in math for her bachelor's degree. As math coordinator and supervisor, she meets the school math team regularly: she meets the whole team once every two months (13 teachers from first to sixth grades). Each week she meets different groups of teachers according to students' ages or levels of study, thus training and supervising the teachers when necessary. Mary's school is a leader in proficiency in math: "We also participated in the final meeting with the mayor in attendance…My students prepared emojis."

Mary describes her work in the staff room: "One day I came to the staff room with a basket full of matches, puzzles, cards, lots of activities options. These are things I always have with me. I sat with the math teachers and talked about math. For a while, I was the teacher and they were my students.

We played math games with the matches and then they had to create more puzzles. Another time we played with dots on cards with beetles on them." Mary tells about the development of a common language, related concepts, the development of strategies, and the enriching discussion that emerged from the shared learning. Mary: "I have [also] trained pupils from the sixth grade to lead the first grade. They think and explain better than we do." Mary closes the interview by referring to her team. She regards her team as extremely talented and qualified. She is convinced that she could not have succeeded without them.

## Irene's story

At the start of the interview, Irene related that similar to Mary, Adam phoned her and then they met to talk about the scaling up phase of the program.

Irene: I participated in the training program for in-service math teachers. I loved the program. It is very important to develop both mathematical and spatial thinking. [Our pupils] don't try to think by themselves. Adam called me and asked that I lead the program in my school team. I introduced the program to the math teachers. One of them had also participated in the training program. She collaborated with me and [it was good]. I focused on the first grade. For some of the teachers, it was completely new." During the interview, Irene said that she didn't follow up on every single teacher's classroom work: "I heard that one teacher worked only with the best pupils. Others held discussions with the children in a class. Most of them [the teachers] introduced the idea of learning math through puzzles to their classes. We [the school team] agreed to meet once every three or four weeks. The meetings took place through Zoom after school hours. I also met teachers one-on-one at their request. It was not easy.

It is important to stress here that, unlike Mary, Irene is not the school math coordinator. During the interview, she was asked about her interaction with her coordinator. Irene: "The coordinator supported the idea that I could lead a change process with the first-grade math teachers. The whole school participated in the math proficiency competition, and I oversaw the first- and second-grade teachers." Irene said that at the very beginning of the school year, she approached the principal and requested her agreement and support for the initiative. She invited Adam to a meeting at the school and the principal approved the project. Irene: "I taught the third grade. Each lesson began with puzzles. The children really liked it. They were already familiar with this type of material from the second grade. There were puzzles in geometry and it connected with the math curriculum very well." Irene said that most teachers preferred to work only with the best pupils. Teachers said that other pupils were afraid of failure and chose not to participate. Her approach was different: "All children experience success. They succeed in solving problems. Even children who usually don't participate took part. It is a positive experience and I hope it also brings them more success and helps them achieve more." Irene also talked about Adam's guidance: "I met him three times alone and twice with the whole group [of teachers]. He was very supportive and helped me to continue."

At the end of the interview, Irene recommended ongoing training for teachers at school. "Perhaps Adam can give a lecture to the whole math team at the beginning of the school year." She understands the need to recruit the whole team in each school to reach the best outcomes.

What prevented the other teachers from leading the change?

It is important to stress here that the three teachers who did not lead a change in their schools, continued to teach math in their classrooms using Kanga-Kids' materials and other puzzles, thus activating a STEM pedagogy.

Emily: I participated in the in-service training two years ago. It was very good and I really enjoyed it. It was just me and the teaching/learning materials. I do not need to supervise other people. I liked the program and I implemented a lot of the ideas. My pupils loved the new activities. I worked with the puzzles once a week in the classroom environment. Then Adam asked me to join the next phase…I have never succeeded in leading other people. I'm good at self-management. That's the reason I have worked as a teacher for more than 30 years and never tried to cope with an administrative position or wanted to be in charge." So why did she agree to lead the program at school? Emily: "I agreed because I wanted all children in first-, second-, and third-grades to enjoy the program and experience developing mathematical thinking. Adam really wanted me to do that, and he wanted to meet once a month on Zoom. I know some teachers had succeeded but others had not. One of those who participated is my coordinator and she is used to leading. I met my colleagues [two first-grade math teachers and two-second grade teachers]. I explained how to work with the puzzles, how to develop the ideas and the concepts, stimulating questions, but I didn't continue. I worked [only] with my students." Emily did not try to communicate with the math coordinator in the school. She thinks that the coordinator would surely have supported the initiative if she had known about it. During the interview she related that Adam had suggested coming to the school to meet with the principal and the math coordinator—perhaps even a meeting with the whole math team—but Emily had said that there was no need for his visit. Now she thinks perhaps it was a mistake not to invite Adam to introduce the program at school. By the end of the interview, Emily suggested: "Maybe I could invite Adam to a meeting with the principal, the coordinator, and the math teachers. He could enthuse them with his passion for the program.

An additional two teachers were asked to conduct the second phase of the program: the scaling up. One was interviewed by phone and the other did not want to collaborate with our learning phase. The phone interview was very short, and the teacher said she did not know what to do with the idea of leading part of her team at school.

Obviously, the answers to the research questions are present in the participants' answers.

1. The basic assumptions about the scaling-up program phase were naïve: Adam believed that the basic motivation that brought teachers to the training program, their enthusiasm and collaboration during the meetings, and the support given at individual and group levels should lead to dissemination of the program, but, as Bodilly (1998) claims, designs, by themselves, cannot transform schools, "Schools [can] not simply open an envelope with design specifications inside and transform themselves" (p.11). As has been shown, scaling up is an extremely complex process where even the ecosystem representatives were needed to make it work.

2. The program succeeded in collaboration at two schools. In accordance with Rosenfeld et al. (2006), we analyzed what made this possible: the two teachers were proactive and recruited the principal and math coordinator and collaborated with the teachers in their schools. They also knew how to attain advice and support from Adam and the ecosystem and had participated in additional activities at local (municipal) and national levels. They both set up an agenda and made sure that the regular meetings with teams were part of a wider range of other activities.

3. Two graduates of the program felt competent for the task. They were either a priori in a prominent position (Mary) or had established the leading position (Irene) by forming collaborations with Adam, the principal, the math coordinator, and the teachers and by involving the whole school.

4. Only in these two schools that had involved the whole school in the program did the program graduates succeed in changing teachers' attitudes and routines.

## Discussion and conclusions

This paper presents an attempt to scale up an innovative program for STEM studies. We have shown what worked and what did not. Several questions arise from this trial. As Cooley (2016) puts it, the scaling up phase should rely on three steps:

Step 1: Developing a scaling up plan

Step 2: Establishing the pre-conditions for scaling up

Step 3: Implementing the scaling up process

Step 1: The scaling up plan relied on the idea that professional support at the individual and group levels, together with the basic training conducted in Kanga-Kids, could be the basis for its dissemination. Adam's plan was to recruit the teachers and give them support.

Step 2: The preconditions for participation were success as teachers and having initiative, as stressed by Adam in his interview. The preconditions seemed logical and were seen as potentially contributing to success, but they also revealed the need for additional preconditions related to leadership skills and motivation.

Step 3: The implementation phase was similar to the Kanga-Kids training at all levels of children's education. Individual and group support and guidance were provided throughout the year. It became evident that simply loving the program and being a good teacher was not enough to ensure scaling up the change.

What can be learned from this?

First, as part of our research, we suggest two additional steps for the scaling-up process.

The fourth step should include a continuous process of evaluation and assessment, providing ongoing feedback to all stakeholders. This feedback could inform necessary adjustments and actions.

The fifth step should involve expanding upon the circle, similar to the "Adapt Strategy and Maintain Momentum" step in the Management Framework for Practitioners (Cooley, 2016). This could involve viewing the entire program within the context of the municipal ecosystem and clarifying its role in the dialogue between implementation in each school and the overall concept.

As Bodilly (1998) noted, many variables are involved in the scaling-up process, including teacher selection, school climate, design and team factors, school structure, and leadership skills. Recognition and appraisal of the program at the governmental level are also necessary for mainstream change.

Bodilly (1998) proposes the scaling-up phase as a long-term process that should be evaluated and assessed for at least three years until the completion of its implementation in the school, while Cooley (2016) suggests that a national scaling-up process takes at least 15 years. Our research indicates that one variable that was not adequately considered is the level of preparedness of teams and course graduates for the scaling up phase, which should be verified as a precondition for participation. Leadership skills are also essential for success and should be tested or taught as a precondition.

The scaling-up process has just begun, and this research provides the program and its creators with an opportunity to learn and draw conclusions. As McLaughlin and Mitra (2001) suggest, deeper theoretical exploration and practical trials, as proposed by Cooley (2016), could help address the challenges of scaling up. Policy support and working toward mainstreaming are also necessary, as noted by Bodilly (1998) and the European Commission (Barnett, 2022).

To conclude, it should be stressed that this is an ongoing study, and more fieldwork and research are needed.

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## Section 5

# MATHEMATICS & ARTS & LANGUAGE

# Connecting arithmetic and geometry as an artistic expression of collateral creativity

Sergei Abramovich[1] and Viktor Freiman[2]

*The goal of this paper is to discuss the notion of collateral creativity in the context of instrumental integration of symbolic and visual mathematics as a creative art. Collateral creativity is defined by the authors (Abramovich & Freiman, 2022) as an accidental but favorable outcome of problem solvers' hidden ideas of mathematics through technologically supported pedagogic mediation. The concept of instrumental act was introduced by Vygotsky (1930) to highlight appropriate uses of tools as means of reconstruction of the whole structure of one's behavior in the process of problem solving. Notably, the idea of mathematics as a creative art was discussed by Halmos (1968) who, in particular, saw both mathematics and painting as having origins in "physical reality."*

*Keywords: Arithmetic, geometry, collateral creativity, artistic expressions, mathematics.*

## Introduction

Halmos states: "The origin of painting is physical reality, and so is the origin of mathematics...[which] is a creative art because mathematicians create beautiful new concepts" (p. 388). Consider an example of collateral creativity by a second grader who was given eight square tiles and asked by a teacher to construct a rectangle using all the tiles (Abramovich, 2019, p. 90). Whereas the traditional expectation of the teacher included two rectangles as eight can be decomposed in two factors only (without regard to their order), the student constructed a square with a hole in the middle shown in Figure 1. Three educational aspects of this classroom episode can be identified. First, the construction was collateral to the number of tiles—while two rectangles can be constructed out of ten tiles as well, a rectangle with an internal hole may not be constructed using ten tiles. That is, due to the number of tiles given, the construction was accidental, yet this educative outcome was favorable as the student was praised by the teacher rather than being told (thus denying any creativity) that rectangles do not have holes. Second, the instrumental nature of this construction is in the use of manipulatives, which allowed the student to reconstruct the traditional structure of behavior when using a marker to draw rectangles on the white board. Third, the construction of the rectangle with a hole had origin in the concrete physical situation and its perceived aesthetic pleasure allowing the student to experiment with tiles by introducing an artistic expression and design thinking (Avsec, 2021) into a physical experiment. A competent teacher can recognize numeric properties of geometric characteristics of this rectangle the area of which, numerically, is half of the perimeter (including the hole). For instance, the value of the area is half of the value of perimeter (external part = 12 + inner perimeter = 4). In other words, visual and symbolic mathematics go hand-in-hand connecting geometry and number theory.

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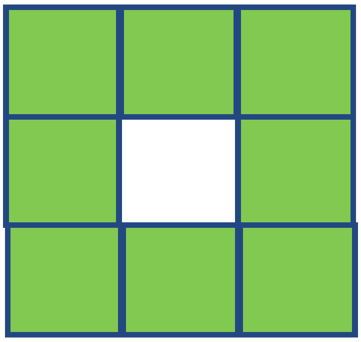

**Figure 1: A rectangle with a hole**

Examples we will discuss in this paper aim to lead to appreciation of collateral creativity as students' artistic expression of mathematical beauty. For instance, when exploring coloring a map with Grade 2 students, one of the authors was struck by the following example of a map that requires four colors provided by one of his students (Figure 2).

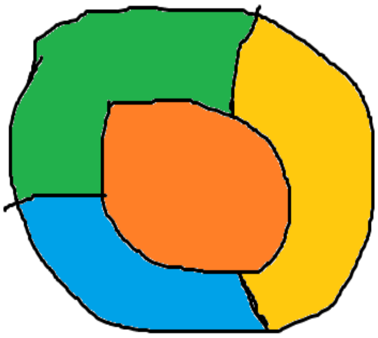

**Figure 2: Four regions—four colors**

This example does not only surface another masterpiece of students' creativity but also highlights the importance of fostering it from early grades. Another example from the same classroom echoes Kolmogorov's well-known insight, which we will discuss later in the paper to deal with a property of decomposing a square into a sum of odd number; this time one of the students was exploring creating patterns with unit cubes. They finally found the following construction exclaiming that adding 3 squares to 1; 5 squares to 4; 7 squares to 9, etc. always produces a square (Figure 3).

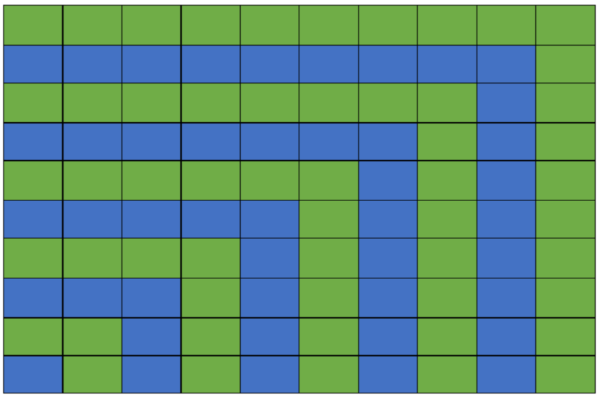

**Figure 3: Constructing squares by a second grader**

One more example of an activity with the same material (unit cubes) comes from Grade 1 students who were asked to build squares on the sides of different types of triangles (obtuse-angle, acute-angle, right-angle) drawn on a piece of paper. One of these triangles had sides 3, 4, 5. After completing the squares, one student remarked that the sum of the number of cubes built on two of the sides is equal the number of cubes built on the third side, an observation that led the child to discussing what was special in this triangle intuitively grasping a sense of the relationship between the sides of a right triangle.

In the next sections we provide additional examples to demonstrate how collateral creativity could be an outcome of students' artistic expression while showing how teachers could support a deeper mathematical investigation upon it.

## Connecting visual and symbolic

One of the problems in the theory of numbers deals with representing numbers as sums of other numbers. Such sums are referred to as additive decompositions of integers. For example, there are several additive decompositions of the number 25 each of which can be associated with the history of mathematics. These include the decomposition of 25 in five consecutive odd numbers (25 = 1+ 3 + 5 + 7 + 9), something that at the pre-school age was noticed by A. N. Kolmogorov, one of the major contributors to mathematics of the 20$^{th}$ century. Kolmogorov considered patterns of that kind as his first mathematical discovery (Tikhomirov, 2001). Another decomposition of 25 is in two consecutive squared integers (25 = 9 + 16)—a special case of the Pythagorean theorem, from where the most famous Pythagorean triple, (3, 4, 5), follows. As an aside, note that just like the rectangle with a hole (Figure 1), the Pythagorean triangle with the side lengths 3, 4, and 5 also has area that is numerically half of its perimeter. That is, this relation could be recognized by a student in a collaterally creative fashion had the teacher shared with the students a similar relation regarding the rectangle with a hole. Historically noticeable is decomposition of 25 in two consecutive triangular numbers (25 = 10 + 15), an observation used in the 18th century by a Dutch minister of church and mathematics teacher Élie de Joncourt to compute squares and square roots (Roegel, 2013). Number theorists would mention decomposition of 25 in three prime numbers (25 = 3 + 5 +17), a special case that was only recently proved in the Ternary Goldbach Conjecture (Helfgott, 2014) which, as mentioned in Vavilov (2021), was first formulated by Descartes. Finally, 25 is a perfect square, the name borrowed from plane geometry as the sum 1 + 3 + 5 + 7 + 9 can be rearranged into another sum of five integers, yet all equal (pointing at the shape of square), 25 = 5+ 5 + 5 + 5 + 5 $= 5 \times 5 = 5^2$, so that the equality 1 + 3 + 5 + 7 + 9 = 5+ 5 + 5 + 5 + 5 can be interpreted as a split of square into five gnomons.

As a way of connecting arithmetic and geometry a task of partitioning a shape (the whole) into several parts (the fractions) can also be formulated (although historically, geometric images appeared before their symbolic descriptions were developed). Over the centuries, the development of mathematical knowledge evolved by considering the primordial nature of concrete objects including geometric shapes over the secondary nature of words and other signs that describe specific combinations and properties of those objects. Following Vygotsky (1978), one can say that mathematical knowledge had been developed through the transition from dealing with the "first-order symbols...directly denoting objects or actions...[to] the second order symbolism, which involves the creation of written signs for the spoken symbols of words" (p. 115). Through such partitioning, different shapes can be created and using visualization as support for rigor and motivation for conceptualization, a great deal of mathematical ideas can be discussed. Johann Heirich Pestalozzi (1746–1827)—a Swiss educational reformer—argued that visual understanding is the foundation of conceptual thinking and he encouraged “the children to draw angles, rectangles, lines and arches, which he said constituted the alphabet of the shape of objects, just as letters are the elements of the words” (Arnheim, 1969, p. 299). In the age of computers, this pedagogical idea of the early 19th century can be enhanced by using a dynamic geometry application when computational experiments, based on the ability of a digital tool to interactively measure different characteristics of geometric shapes, motivate conjecturing and serve as a window to formal justification of technology-motivated conjectures.

Number theory, with its origin in symbolic description of images found in real life, includes problems dealing with additive decomposition of integers. Over the centuries, the development of mathematical knowledge evolved by describing the properties of geometric shapes in a symbolic form. For example, the number 25 can be decomposed in the first five odd numbers (alternatively, the corresponding square can be split in five gnomons (L-shape)), in two consecutive triangular numbers (making a square from two isosceles triangles), and in the sum of two squares of side lengths 3 and 4. In the classroom, all those connections between symbols and images can be instrumentally presented by using manipulatives as well as digital tools. In the authors’ experience, such artistic presentations of mathematical ideas motivate students to ask collaterally creative questions. In the specific context of decomposition of numbers and corresponding images, the importance of the center of a square emerges and the task of partitioning square into equal parts sharing the center can be formulated and explored. In this paper, we will provide an example of a simple algorithm of partitioning a square into any number of equal parts sharing the center thus leading to various aesthetic representations.

## Center as symbolic and visual characteristics of a mathematical structure

In all five additive decompositions of the number 25, one can single out the number 5 as the center of the decompositions. This can be demonstrated through the diagrams of Figures 4–7. In Figure 4, the triangular shape formed by the sum of the first five odd numbers is transformed into the sum 5 + 5 + 5 + 5 + 5 with the number 5 being at the center of the sum (1).

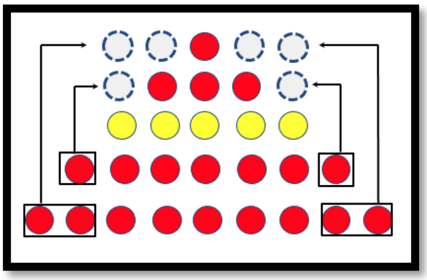

**Figure 4: The sum of the first five odd numbers as a square**

In Figure 5, the two squares represented by 16 and 9 counters are also rearranged into sum (1).

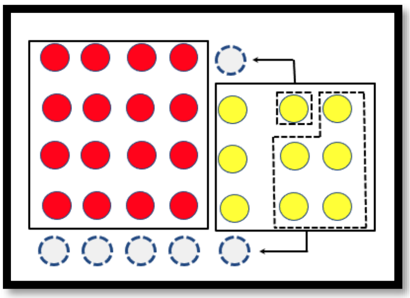

**Figure 5: The sum of two squares as a square**

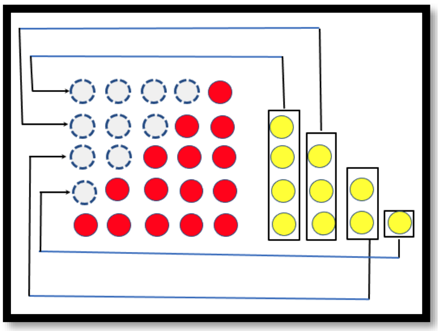

**Figure 6: The sum of two consecutive triangular numbers as a square**

In Figure 6, the sum of two triangular numbers represented through right triangles is also rearranged into sum (1). One may note that the original arrangement of counters in Figure 5 resembles that of Figure 6, yet the transformation was done differently. At the same time, the transformation carried out in Figure 4 shows how the sum 1 + 2 + 3 + 4 can be found as $\frac{4(4+1)}{2}$ and generalized to the form $1+2+3+\cdots+n=\frac{n(n+1)}{2}$. In particular, this shows how the diversity of visual representations provided by concrete materials affects mathematical symbolism of algebra.

Finally, in Figure 7, the sum 3 + 5 + 17 of three prime numbers is also transformed into sum (1), which is a special case that, as noted, was only recently proved in the Ternary Goldbach Conjecture (Helfgott, 2014).

This motivates considering the center of a shape as an important element of the shape. Many mathematical problems in school geometry deal with the construction of their centers, however the meaning of the word center is understood. In regular shapes, like equilateral triangle, square, regular pentagon, and so on, the center belongs to the intersection of different elements of the shapes, and it can be used to subscribe or circumscribe a circle around the shape. In what follows, the square will be considered as well as its decomposition in a given number of parts of equal area.

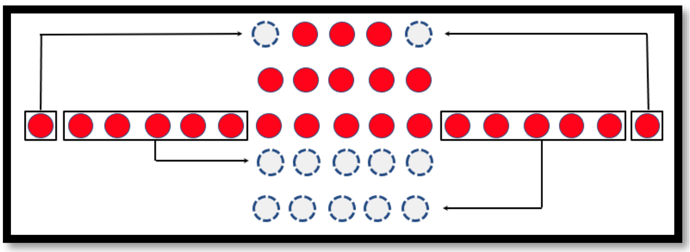

**Figure 7: The sum of three prime numbers as a square**

**Remark 1.** Of course, not every square constructed out of counters has a visually identifiable square. But as always in mathematics, visualization opens a window to abstraction. For example, in $4 \times 4$ square (or in the sum 1 + 3 + 5 + 7) there is no counter as the center of symmetry (no number in the middle). In that case, the center has to be defined proceeding from numeric concreteness and visual affordance provided by the odd number of side lengths or addends. Visually, the definition may include the midpoint of a diagonal of a square (used below in the related geometric constructions); numerically, the arithmetic-mean of two midterms of a sum.

## Dividing a square into five equal parts sharing the same center

The emergence of the center of a polygon as an important entity of geometric constructions, provides many opportunities for dynamic investigation and eventual emergence of collateral creativity by appealing to a visual aspect of hidden relationships (Figure 8).

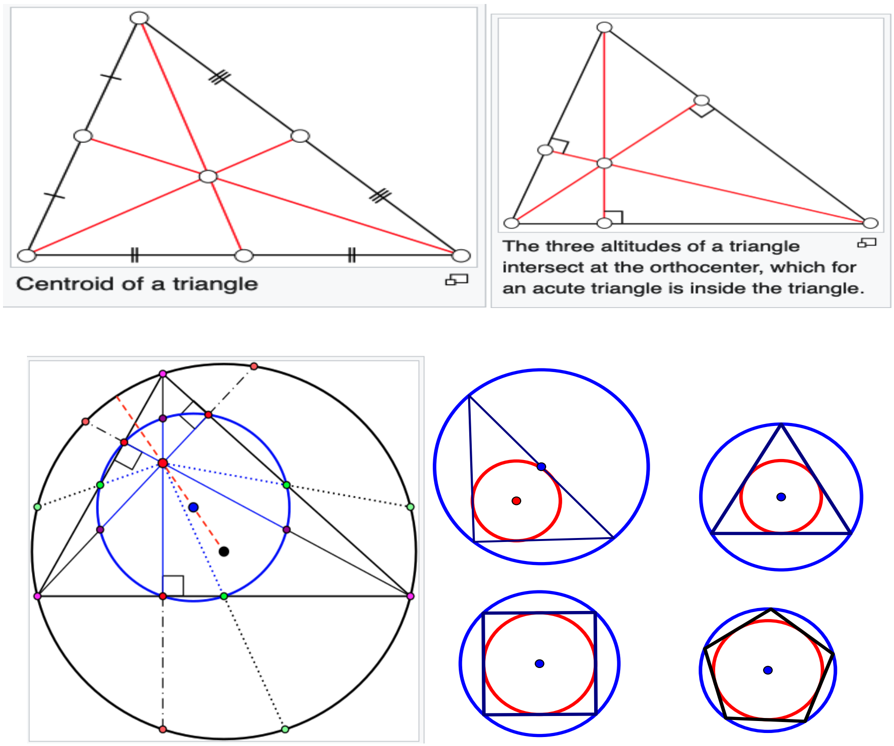


**Figure 8: Visual appeal of the center of a geometric figure**

Indeed, Gerson et al. (2018) mentioned a possibility for a student to find surprising solutions or a new way of looking at a problem; engage in an active process of creating mathematical ideas; conduct an inquiry using technological tools; discuss, and question.

In demonstrating visual images of fractions by connecting arithmetic to geometry, learners of mathematics might be very creative without actually recognizing their creativity. For example, as discussed in Abramovich and Brouwer (2007), an elementary teacher candidate drew a sketch similar to the one shown in Figure 9. The teacher candidate claimed that regions between two consecutive squares are equal to the area of the smallest square. This, in turn, divides the largest (dotted lines) square into five equal parts (four frames and the smallest square), which makes the surface of four 'frames' (space between two consecutive nested squares that share the same center) and the surface of the small square in the middle the same thus creating five equal parts of the large square (the one marked by dotted lines). The teacher candidate did not provide a justification for the claim or built on this creative idea. In this example, the teacher candidate demonstrated collateral creativity as an accidental outcome of working on five squares that share a center point. It was the instructor who identified this as a manifestation of collateral creativity and followed up by posing the problem with the whole class. This is elaborated below. A variety of activities appropriate for elementary, secondary, and tertiary levels of mathematics education stemming from the collaterally creative example of one-fifth shown in Figure 9 can be found in Abramovich and Brouwer (2007).

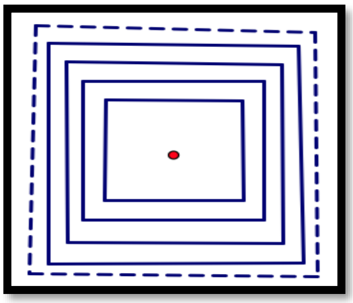

**Figure 9: Collateral creativity of dividing square into five equal parts**

## Collateral creativity as problem posing in the zone of proximal development

An important distinction between creativity and collateral creativity is that the former does not require a teacher in the process of problem solving. The latter, however, does require a competent teacher capable of making the transition from knowing mathematics to using this knowledge in recognizing that a student, in fact, has posed a problem, and then share this recognition with other students. Collateral creativity of a student as was shown in the two examples above—the creation of rectangle with a hole and the construction of equal area parts of a square in the form of frames—without the presence of a "more knowledgeable other" (Vygotsky, 1978) may go unnoticed or just be rejected by the instructor. However, both examples open a window to new mathematical ideas and may be recognized as unintended problem posing by a student.

In general, the activity of problem posing has been a useful method of encouraging the advancement of mathematics and mathematical education for a long time. Problem posing goes back to the 15$^{th}$ century Italy when the first printed book on arithmetic written by an unknown author included problems aimed at explaining how to solve problems appearing in the context of trade. In the 17$^{th}$ century, many non-mathematicians interested in gambling posed mathematical problems aiming to become more informed gamblers. In the keynote address to the 1900 International Congress of Mathematicians, Hilbert (1902) formulated 23 problems from different branches of mathematics, thereby charting the program of mathematical research for the 20$^{th}$ century. Finally, a few years ago, a book on the use of problem solving in mathematics education was published (Singer et al., 2015).

The kind of unwitting problem posing manifested by a second grader (rectangle with a hole) and by an elementary teacher candidate (portioning rectangle into frames of equal areas) may be considered as problem posing stemming from doing mathematics in the zone of proximal development (Vygotsky, 1987). This zone can be described as a dynamic characteristic of cognition that, in a problem-solving situation, measures the distance between two levels of one's development as determined by independent and assisted performances. The construction of a rectangle with a hole by a second grader was done independently.

The child was collaterally creative due to the use of square tiles and in order to recognize his performance as posing a problem—finding the relationship between area and perimeter of rectangle with a hole—requires competent assistance of a teacher. This competence includes a teacher's ability of recognizing the significance of the student's independent performance, sharing this recognition with the student (who, otherwise, would remain in the zone of proximal development) and inviting the entire class to explore the problem posed as a result of collateral creativity by one of their peers. As was mentioned above, the construction of a fractional part of a square was not followed by a justification on the part of the teacher candidate, and it was the candidate's professor who recognized in the construction a new problem-solving milieu well beyond the elementary level. For example, the following problem was posed: How can one partition a square of side $\sqrt{n}$ into $n$ regions of unit area sharing a center?

## Dividing a square into five equal parts sharing the same center

A problem of dividing a 5×5 square into five equal parts sharing the center of the square (constructed as the midpoint of a diagonal) can be solved in several ways. First, using paper and pencil one can find out that it is not possible to have five triangles of equal area sharing the center. Therefore, at least one of the parts has to be a quadrilateral. This raises a new question: Can we have one quadrilateral and four triangles, all of the same area? Such explorations would result in the conclusion that we need three quadrilaterals and two triangles. A possible partition is shown in Figure 5 where two adjacent sides of the square are divided in the ratio of 1 to 4 and another two sides in the ratio 2 to 3. This understanding of how partitioning the sides of the square in two segments can be done can be developed experimentally through trial and error by using dragging and measuring features of dynamic geometry software. Then, the center of the square (the midpoint of a diagonal) has to be connected with one vertex and with the corresponding points of division of the sides through which three triangles and two quadrilaterals have been created. The functionality of the *Geometer's Sketchpad* allows for an experiment, and *computational demonstration, using trial and error,* help show that the five parts do indeed have equal areas (Figure 8). This, however, does not mean that the tool "abolishes and makes unnecessary a number of natural processes" (Vygotsky, 1930), such as the process of formal mathematical proof. Rather, the tool sets the stage for a discussion of why the parts have equal areas and thus it *coordinates the course* of mathematical behavior by a student. In other words, proceeding from the computational demonstration, a student can be led to some formal geometric investigations structured by a set of questions requesting explanation. This first question (also seen as posing a problem formulated within the zone of proximal development of at least some learners of mathematics) is: Why do the five parts have equal areas?

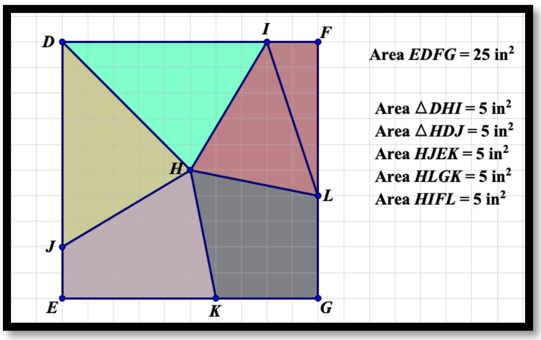


**Figure 10: Dividing a 5×5 square into five equal parts**

To answer this question, note that we have two equal area triangles (having equal in length a base and the corresponding height, respectively) and three quadrilaterals (Figure 10). Using the basic formula, the area of a triangle is equal to the product $\frac{1}{2} \cdot \frac{5}{2} \cdot 4 = 5$ (square units). In order to find the area of a quadrilateral KHLG two equal sides and the diagonals of which form the right angle, the lengths of the latter pair have to be calculated. This would allow one to find the area as half the product of the diagonals (which form the right angle because the medians of isosceles triangles KHL and KGL meet at the same point thus forming the diagonal $HG \perp KL$). We have HG = $\frac{5\sqrt{2}}{2}$ and KL = $2\sqrt{2}$ . Therefore, the area of the quadrilateral KHLG is equal to the product $\frac{1}{2} \cdot \frac{5\sqrt{2}}{2} \cdot 2\sqrt{2} = 5$ (square units). This makes it possible not to calculate areas of equal area quadrilaterals IFLH and JEKH and to subtract 15 from 25 to have 10 square units for both quadrilaterals. This proves that all five parts into which the square EDFG was divided have the same area, 5 square units.

These calculations can be generalized to the case of square of side length $a$. The area of a triangle is the product $\frac{1}{2} \cdot \frac{a}{2} \cdot \frac{4a}{5} = \frac{a^2}{5}$. The diagonals of the quadrilateral with a pair of equal sides and diagonals forming the right angle are $\frac{a\sqrt{2}}{2}$ and $\frac{2a\sqrt{2}}{5}$ . Thus, the area of each quadrilateral is equal to the product $\frac{1}{2} \cdot \frac{a\sqrt{2}}{2} \cdot \frac{2a\sqrt{2}}{5} = \frac{a^2}{5}$. This proves that all five parts into which the square of side length $a$ was divided have the same area, $\frac{a^2}{5}$ square units.

## Dividing a square into seven equal parts sharing the same center

In order to partition a square into seven equal parts that share the center of the square, one has to construct a square of side length $a$ and divide its sides starting from the top-left corner and going clockwise as follows:

$$\left(\frac{2a}{7} + \frac{4a}{7} + \frac{a}{7}\right) + \left(\frac{3a}{7} + \frac{4a}{7}\right) + \left(\frac{4a}{7} + \frac{3a}{7}\right) + \left(\frac{a}{7} + \frac{4a}{7} + \frac{2a}{7}\right).$$

By connecting the center of the square with the points into which its sides have been partitioned, we see four triangles of base $\frac{4a}{7}$ and height $\frac{a}{2}$ as well as three quadrilaterals one of which has mutually perpendicular diagonals of lengths $\frac{a\sqrt{2}}{2}$ and $\frac{2a\sqrt{2}}{7}$ . Therefore, the sum of the areas of those four triangles and the quadrilateral is equal to

$$4 \cdot \frac{1}{2} \cdot \frac{4a}{7} \cdot \frac{a}{2} + \frac{1}{2} \cdot \frac{2a\sqrt{2}}{7} \cdot \frac{a\sqrt{2}}{2} = 4\frac{a^2}{7} + \frac{a^2}{7} = \frac{5a^2}{7}.$$

The joint area of the remaining two equal area quadrilaterals is equal to $a^2 - \frac{5a^2}{7} = \frac{2a^2}{7}$

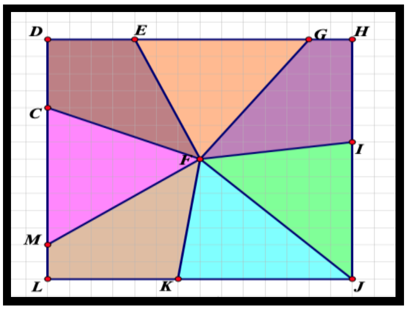


**Figure 11: Dividing a square into seven equal parts**

## Dividing a square into nine equal parts sharing the same center

In order to solve this problem, consider first an $a \times a$ square divided into five equal parts. In order to divide this square into nine equal parts, each side a triangle has to be adjusted so that the sides are divided beginning from the top-left corner as follows:

$$\left(\frac{4a}{9} + \frac{4a}{9} + \frac{a}{9}\right) + \left(\frac{3a}{9} + \frac{4a}{9} + \frac{2a}{9}\right) + \left(\frac{2a}{9} + \frac{4a}{9} + \frac{3a}{9}\right) + \left(\frac{a}{9} + \frac{4a}{9} + \frac{4a}{9}\right).$$

As shown in Figure 12, we have six triangles and three quadrilaterals. All the triangles have the same length of the base, $\frac{4a}{9}$, and the same height, $\frac{a}{2}$; thus, having area $\frac{a^2}{9}$. One has to show that the area of each of the two quadrilaterals with mutually perpendicular diagonals is equal to the area of a triangle. Indeed, similarly to the case of Figure 11, the area of this quadrilateral is equal to the product $\frac{1}{2} \cdot \frac{2\sqrt{2}a}{9} \cdot \frac{\sqrt{2}a}{2} = \frac{a^2}{9}$. Therefore, for the remaining two equal area quadrilaterals we have $a^2 - \frac{7a^2}{9} = \frac{2a^2}{9}$ square units so that all nine parts into which the square was partitioned have area $\frac{a^2}{9}$ (square units).

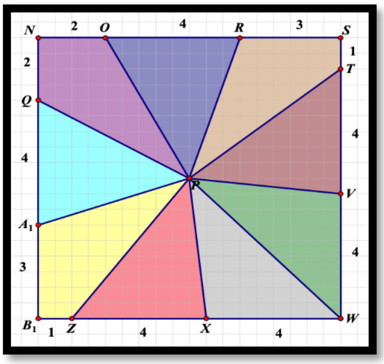


**Figure 12: Dividing a square into nine equal parts**

In general, having a square of side $a$, its sides are divided as follows:

$$\left(\frac{2a}{9} + \frac{4a}{9} + \frac{3a}{9}\right) + \left(\frac{a}{9} + \frac{4a}{9} + \frac{4a}{9}\right) + \left(\frac{4a}{9} + \frac{4a}{9} + \frac{a}{9}\right) + \left(\frac{3a}{9} + \frac{4a}{9} + \frac{2a}{9}\right).$$

The area of each of the six triangles with base $4a/9$ linear units is equal to $a^2/9$. In order to find the area of a quadrilateral with sides $a/9$, $3a/9$, sqrt(130)a/18, sqrt(90)a/18, one has to multiply the area

9 by the factor $a^2/81$ to have area $a^2/9$. Subtracting $8a^2/9$ from $a^2$ yields $a^2/9$ as the area of third quadrilateral with equal sides forming the right angle.

## Dividing a square into 11 equal parts sharing the same center

Similarly, a square can be divided into 11 (Figure 13) equal parts. In that case, a square of side length *a* has its sides divided from the top-left corner in the clockwise order as follows:

$$\left(\frac{4a}{11}+\frac{4a}{11}+\frac{3a}{11}\right)+\left(\frac{a}{11}+\frac{4a}{11}+\frac{4a}{11}+\frac{2a}{11}\right)+\left(\frac{2a}{11}+\frac{4a}{11}+\frac{4a}{11}+\frac{a}{11}\right)+\left(\frac{3a}{11}+\frac{4a}{11}+\frac{4a}{11}\right).$$

The parts include eight triangles, each of which has the base equal to $4a/11$ linear units thus making its area equal to $a^2/11$ and three quadrilaterals. Using the method described above, one can calculate that the area of each of the two quadrilaterals with perpendicular side lengths $a/11$ and $3a/11$ linear units and see that it is equal to $a^2/11$. This implies that the area of the third quadrilateral is also $a^2/11$. Therefore, all 11 parts have the same area.

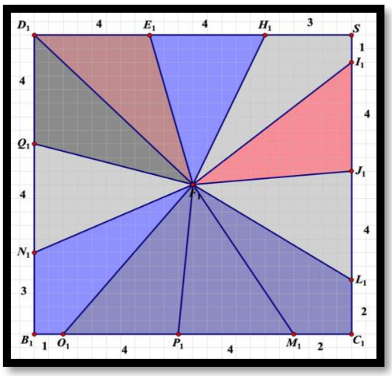

**Figure 13: Dividing a square into 11 equal parts**

## Dividing a square into 13 equal parts sharing the same center

Similar explorations can be carried out when dividing a square into 13 equal parts sharing the same center. As shown in Figure 14, partitioning side length of the square beginning from its top-left corner can be done as follows: (1 + 4 + 4 + 4) + (4 + 4 + 4 + 1) + (3 + 4 + 4 + 2) + (2 + 4 + 4 + 3). In general, in the case of the square of side length *a*, we have the following partition:

$$\left(\frac{a}{13}+\frac{4a}{13}+\frac{4a}{13}+\frac{4a}{13}\right)+\left(\frac{4a}{13}+\frac{4a}{13}+\frac{4a}{13}+\frac{a}{13}\right)+\left(\frac{3a}{13}+\frac{4a}{13}+\frac{4a}{13}+\frac{2a}{13}\right)+\left(\frac{2a}{13}+\frac{4a}{13}+\frac{4a}{13}+\frac{3a}{13}\right).$$

Each of the 10 triangles has area $\frac{1}{2}\cdot\frac{4a}{13}\cdot\frac{a}{2}=\frac{a^2}{13}$.

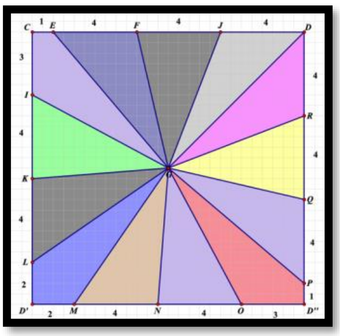

**Figure 14: Dividing a square into 13 parts**

## Partitioning square in 4k + 1 equal parts sharing the same center

Using examples discussed in sections 6, 8, 10, and noting that the numbers 5, 9, 13 are one greater than a multiple of four, that is, are of the form $4k + 1$, a more general problem of partitioning a square of side length $a$ in $4k + 1$ equal parts that share the center of the square can be solved. Due to the equalities

$$\begin{aligned}
&\left(\underbrace{\frac{4a}{4k+1}+\cdots+\frac{4a}{4k+1}}_{k\ times}+\frac{a}{4k+1}\right)+\left(\frac{3a}{4k+1}+\underbrace{\frac{4a}{4k+1}+\cdots+\frac{4a}{4k+1}}_{k-1\ times}+\frac{2a}{4k+1}\right)\\
&+\left(\frac{2a}{4k+1}+\underbrace{\frac{4a}{4k+1}+\cdots+\frac{4a}{4k+1}}_{k-1\ times}+\frac{3a}{4k+1}\right)\\
&+\left(\frac{a}{4k+1}+\underbrace{\frac{4a}{4k+1}+\cdots+\frac{4a}{4k+1}}_{k\ times}\right)\\
&=\left(\frac{4ak}{4k+1}+\frac{a}{4k+1}\right)+\left(\frac{3a}{4k+1}+\frac{4a(k-1)}{4k+1}+\frac{2a}{4k+1}\right)\\
&+\left(\frac{2a}{4k+1}+\frac{4a(k-1)}{4k+1}+\frac{3a}{4k+1}\right)+\left(\frac{a}{4k+1}+\frac{4ak}{4k+1}\right)\\
&=\frac{(4k+1)a}{4k+1}+\frac{(3+4k-1+2)a}{4k+1}+\frac{(2+4k-1+3)a}{4k+1}+\frac{(4k+1)a}{4k+1}\\
&=a+a+a+a=4a
\end{aligned}$$

the square has been partitioned in $4k - 2$ triangles of base $\frac{4a}{4k+1}$ and height $\frac{a}{2}$ as well as three quadrilaterals one of which has mutually perpendicular diagonals of lengths $\frac{a\sqrt{2}}{2}$ and $\frac{2a\sqrt{2}}{4k+1}$. Therefore, the sum of the areas of those $4k - 2$ triangles and the quadrilateral is equal to

$$(4k-2)\cdot\frac{1}{2}\cdot\frac{4a}{4k+1}\cdot\frac{a}{2}+\frac{1}{2}\cdot\frac{2a\sqrt{2}}{4k+1}\cdot\frac{a\sqrt{2}}{2}=(4k-2)\frac{a^2}{4k+1}+\frac{a^2}{4k+1}=\frac{(4k-1)a^2}{4k+1}.$$

The joint area of the remaining two equal area quadrilaterals is equal to $a^2-\frac{(4k-1)a^2}{4k+1}=\frac{2a^2}{4k+1}$.

## Partitioning a square in $4k-1$ equal parts sharing the same center

Using examples discussed in sections 7, 9 and noting that the numbers 7, 11 are one smaller than a multiple of four, that is, are of the form $4k-1$, a more general problem of partitioning a square of side length $a$ in $4k-1$ equal parts sharing the same center of the square can be solved. Due to the equalities

$$\left(\underbrace{\frac{4a}{4k-1}+\cdots+\frac{4a}{4k-1}}_{k-1\ times}+\frac{3a}{4k-1}\right)+\left(\frac{a}{4k-1}+\underbrace{\frac{4a}{4k-1}+\cdots+\frac{4a}{4k-1}}_{k-1\ times}+\frac{2a}{4k-1}\right)$$
$$+\left(\frac{2a}{4k-1}+\underbrace{\frac{4a}{4k-1}+\cdots+\frac{4a}{4k-1}}_{k-1\ times}+\frac{a}{4k-1}\right)$$
$$+\left(\frac{3a}{4k-1}+\underbrace{\frac{4a}{4k-1}+\cdots+\frac{4a}{4k-1}}_{k-1\ \ times}\right)=2\frac{8a(k-1)+6a}{4k-1}=4a\frac{4k-1}{4k-1}=4a,$$

the square has been partitioned in $4(k-1)$ triangles of base $\frac{4a}{4k-1}$ and height $\frac{a}{2}$ as well as three quadrilaterals one of which has mutually perpendicular diagonals of lengths $\frac{a\sqrt{2}}{2}$ and $\frac{2a\sqrt{2}}{4k-1}$. Therefore, the sum of the areas of those $4(k-1)$ triangles and the quadrilateral mutually perpendicular diagonals is equal to

$$4(k-1)\cdot\frac{1}{2}\cdot\frac{4a}{4k-1}\cdot\frac{a}{2}+\frac{1}{2}\cdot\frac{2a\sqrt{2}}{4k-1}\cdot\frac{a\sqrt{2}}{2}=4(k-1)\frac{a^2}{4k-1}+\frac{a^2}{4k-1}=\frac{(4k-3)a^2}{4k-1}.$$

## The four-color problem

Another artistic exploration considered in this paper deals with finding the number of regions into which a square can be divided by connecting specific points belonging to its sides (including vertexes) that were used in partitioning a square into equal parts (Figure 16). This task not only brings learners back to numbers, which can be found in the Online Encyclopedia of Integer Sequences, but it leads to a classic domain of mathematical knowledge known as the Four Color Problem (Appel & Haken, 1977). For example, as shown in Figure 14, whereas two colors are enough to color eight and 14 regions, already 23 regions require three colors. We argue that collateral creativity while being an accidental outcome of classroom activities is nonetheless a favorable one as finding answers to collaterally creative questions leads to epistemic development of both more and less knowledgeable practitioners of mathematics education.

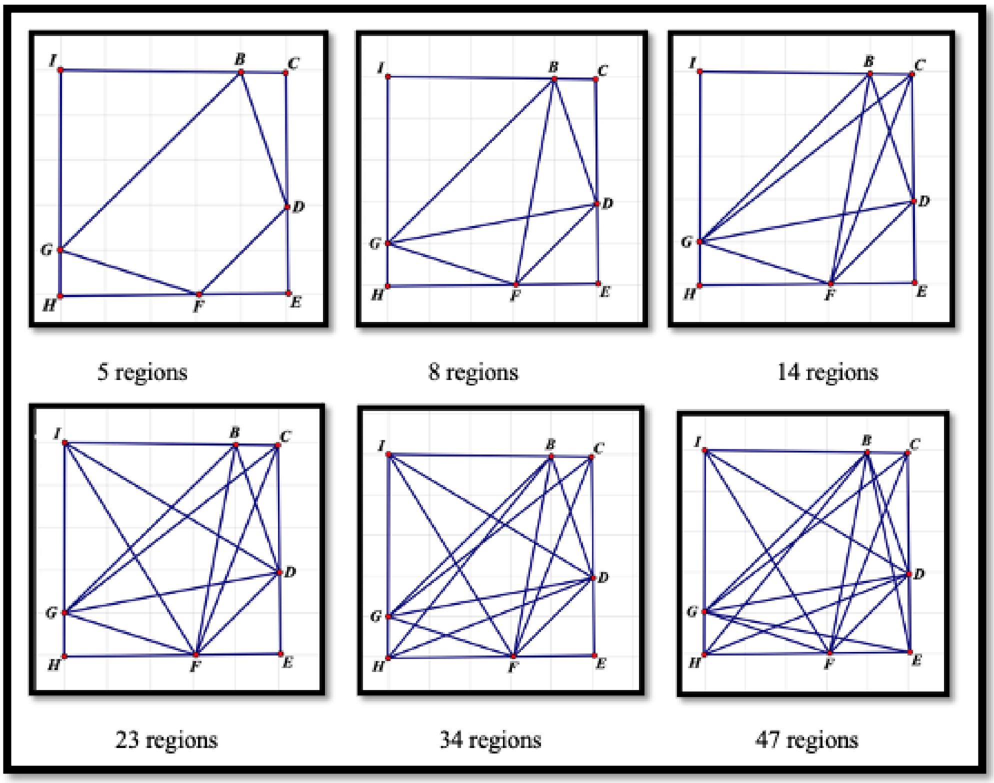


**Figure 15: Counting the number of regions**

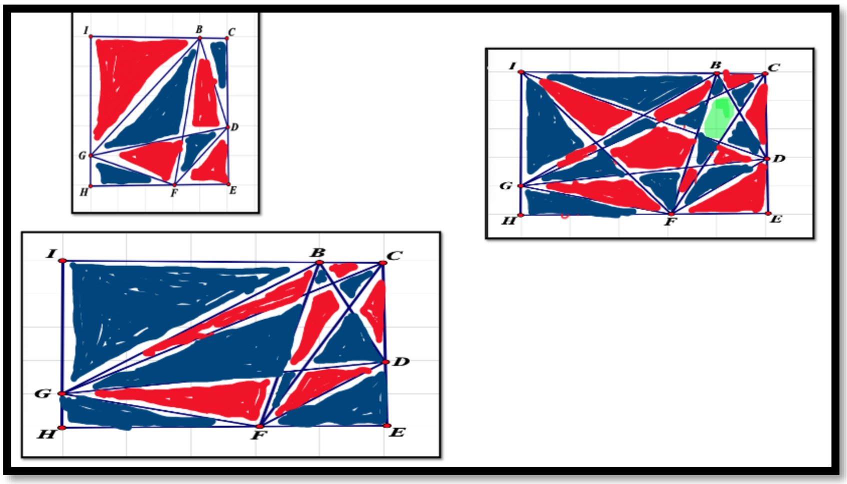

**Figure 16: Three colors are needed for 23 regions**

## Conclusions

In summary, we provided several examples of how mathematics and artistic expression are connected giving boost to collateral creativity:

- Arts have inspired many mathematicians to create new insights into the beauty of patterns and relationships (Halmos, 1968).
- Educators of the past and modern times see in artistic expression a means to visualize mathematics thus making learning more intuitive and dynamic.

- Through these intrinsic connections, new (and unexpected!) ideas can emerge thus opening a door to collateral creativity (Abramovich & Freiman, 2022).

Awareness of this collaterally creative outcome can help teachers to invite students to more advanced and complex mathematical explorations and thinking.

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# Poetry and mathematics in relation

Richard Barwell[1] and Yasmine Abtahi[2]

*We explore the value of poetry for thinking critically about mathematics and education. Although poetry has been proposed as a tool for teaching mathematics or for analysing mathematics classroom data, we are interested in something slightly different. We argue that reading some situations through poetry alongside mathematical readings, the limitations of mathematics become more apparent. We illustrate this idea with two examples drawn from our own ongoing work. One example concerns the management of wolf populations, looked at from the perspective of mathematical modelling and a poem by Ted Hughes. The other concerns the relational nature of mathematical knowing, looked at from the perspective of a poem by Rumi read against the Ontario mathematics curriculum.*

*Keywords: Mathematics education, epistemology, poetry.*

## Introduction

We are critical mathematics educators interested in how different ways of thinking and talking about mathematics and mathematics education have wider effects in the social and ecological worlds that we live in. In this article, we consider how poetry can contribute to this project, based on some of our recent writing. The mathematics education literature does include some writing about poetry, as a metaphor for teaching mathematics (Taylor, 1980), or about poetic linguistic patterns in students' mathematical talk (Staats, 2008), to give two examples. Our goal in this paper, however, is not to discuss how poetry can be a useful tool for teaching mathematics or can be a useful tool for analysing mathematics classroom interactions, although it clearly has potential to be both. Instead, we consider how poetry can help us to better understand relational aspects of mathematics, prompted by the kinds of responses that poetry can provoke. Like Khan (e.g., 2011), we see the potential of engaging with poetry for thinking differently about mathematics and mathematics education from a critical perspective. Khan's project is to produce mythopoetic texts as a way to propose a mathematics curriculum that explores "the pressing issues of grief, trauma and reconciliation" and "its potential for and role in decolonisation, liberation, justice and sustainability" (p. 17). Our interest in this paper is not so much in producing poetic curriculum texts; rather, we reflect on how our engagement with poetry (as readers) opens up new ways of thinking and knowing about mathematics education in relation to social justice and sustainability.

## (In)visible interrelations

Critical mathematics, particularly as represented by the work of Skovsmose (1994), has argued that mathematics has a "formatting" role in society. That is, the language game of mathematics embeds particular logics and assumptions, often through technology, into society and has real effects. O'Neil

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(2017), for example, examines how human-designed mathematical systems produce structural inequality in relation to policing, health, insurance or advertising. In our own work, we have explored how mathematics shapes discourses related to climate change, environmental sustainability, obesity, or social justice (Abtahi, 2022; Barwell, 2018; Hall & Barwell, 2021). We think about these kinds of issues in terms of the sorts of relations that mathematics makes visible or obscures. This is important because, from a critical mathematics perspective, the knowing of mathematics could limit or restrict our experiences and our understanding of the social and environmental world and our place within it.

As mathematicians, we often understand our lives through networks of numbers and calculations. Through mathematics our conceptualisations of our lives and of the world are through a highly quantified framework (Ernest, 2010). Such discourses and ways of seeing and being become a particular controlling component in societies and in states, "in a society in which what matters is what counts or is counted" (Ernest, 2021, p. 12). Mathematics or mathematics-informed discourses may, for example, through mathematical modelling, construct the Earth's climate as a controllable system, and thus a relation in which humans control the Earth (Barwell, 2018). Or mathematical discourses may construct some forms of mathematics as universal, so that a relation of dominance is constructed between people who have access to these forms and those who do not (Abtahi, 2022). We have both written about the problematic way in which such relations are implicit and uninterrogated.

In this paper, we explore how poetry has helped us to interrogate some of these relations and that would otherwise have been unnoticed. We ask how engaging with poetry may influence our work as mathematics education researchers, especially with respect to challenging questions about equity, epistemology, or ecological collapse. For us, the exploration of the usefulness of poetry has less to do with the specific messages conveyed by the poetry. We are interested in how the poems provided us with alternative stories and ways of thinking that can draw us into different forms of attention and awareness. Poetic thinking, for example, may include dimensions such as linguistic metaphor, polysemy, ambiguity and emotional force. While these dimensions may also be present in mathematics, they are not necessarily foregrounded in situations in which mathematics is used to make sense of the world. Hence, poetry offers alternative ways of thinking that we seek to set in relation with mathematics.

In what follows, we first conceptualise in what ways providing alternative stories and ways of thinking is not only important but also necessary for understanding the types of relationalities that are constructed by mathematics. We then give two examples of our experiences with using poetry to think differently in mathematics education.

## Alternatives and stories: Truth versus relations!

From the perspective of Skovmose's critical mathematics education, we make decisions using mathematics, often without entirely realising, and certainly without paying much attention to the effects that mathematics has in organising our decision-making space. For example, individuals make decisions about their lives based on their body mass index (Hall & Barwell, 2022). They may choose to change their diet or lifestyle, perhaps based on advice from a medical practitioner. Governments make decisions about energy policy based on mathematical models of the climate (Barwell, 2018). And teachers make decisions about what to teach based on standardised test results and prevailing ideas about what mathematics is. But these decisions can all have harmful consequences (see, for example, Abtahi, 2022).

In our previous work (and that of many other critical mathematics educators), we have become aware of how mathematics is implicated in structuring hierarchies of different kinds. For example, the body mass index is implicated in hierarchies of desirable body types, climate models are implicated in an anthropocentric response to climate change, and the 'standard' form of mathematics taught in schools is implicated in the perpetuation of settler colonialism in Canada. These hierarchies are constructed, problematic, forms of relation, but others are possible.

More specifically, mathematics is implicated in constructing a hierarchical relationship between humans and other species, derived from epistemological assumptions that construct mathematics as a tool for intervening in and operating on the ecosystem, even in many cases as part of efforts to undo the disastrous effects of human activity on our planet. These assumptions are connected to a deep late-modern cultural understanding of humans as being distinct from Nature (Morton, 2010), within a narrative of human exemptionalism (Bowers, 2001; Catton & Dunlap, 1980). This hierarchical relationship is linked to others that produce racism, sexism, classism and so on, all of which can be traced back in part to the mathematisation of different aspects of the social-ecological world (Martusewiczet al., 2014). This analysis has led to rethink how mathematics is part of relations between humans and other species (Gutiérez, 2017), including from a dialogic perspective (Barwell, 2022; Barwell et al., 2022).

Equally, mathematics constructs specific hierarchical relationship between some humans and some other humans.  This construct also carries an epistemological assumption. That is, the assumption that mathematics of a group of people are more *real* mathematics than mathematics of another. Abtahi (2022) notices this hierarchical relationship in her teaching of Canadian provincial-mandated mathematics to a group of Canadian indigenous colleagues. Reflecting on her teaching, she explains that although she deployed her cultural resources in the process of teaching the mathematics and similarly encouraged her students to do so, she still limited them to acquiring the kinds of knowledge that was counted as mathematics in the Ontario mathematics curriculum. She says: "In my knapsack, I carried fragmented and compartmentalised sets of knowledges, with little to no regard for any particular epistemologies, ways of knowing" (p. 157) specifically of my Canadian indigenous colleagues. Consequently, she sets out to explore the kinds of harm she had caused, through the type of mathematics that she taught and through the type of curriculum that she used.

For Rorty (2006), the "best answer" to the question of how to make better decisions "is that individuals become aware of more alternatives. The source of these new alternatives is the human imagination. Rorty says that imagination "is the ability to come up with new ideas, rather than the ability to get in touch with unchanging essences" (p. 372). This is exactly why poetry has become important for our work in mathematics education research (again, see also Khan, 2011). By thinking with poetry alongside mathematics, we are working with two rather different ways of knowing at once, which prompts our imagination: in particular, it brings to light alternatives to mathematical knowing which might turn out to be valuable.

One feature of mathematical discourses is that they are highly naturalised; it is difficult to see how they could be different, or, to say it another way, they create blind spots. Poetry has helped us to attend to relationalities which previously have been in our blind spots. Poetry has also given us an understanding of the sorts of cruelty that humans are capable of, and which mathematics makes invisible. That is why we believe that seeking alternatives to understanding the world with mathematics might help to make visible what has been made invisible through mathematics.

In summary, we are both very aware of how mathematics can be implicated in problematic relations (although we should be clear that these relations are made by humans, not by some externally existing mathematics). At the same time, we do not rely on mathematics as our only way of making such relations. In particular, we have recently reflected on how poetry offers us a way into different kinds of relations, which in turn allows us to reflect on the blind spots of (our) mathematics. To illustrate these reflections, we offer two examples from our ongoing work.

## Richard's example: Wolf culling

I recently read about a wolf cull in the US mid-West[4]. I found the report upsetting, but was also struck by how mathematics was present, such as in information about the "management" of the wolf population. The report prompted me to return to a poem by Ted Hughes (1989), called *Wolfwatching*. I have been working on a piece of writing that weaves responses to the poem with analysis of the mathematical discourses apparent in scientific or political texts.

In scientific writing, I noticed that mathematical discourses (equations, graphs, tables, descriptions of methods of estimating wolf populations, mathematical models), render wolves as anonymous variables, which feed into broader ideologies in which wolf populations can and should be "managed" in ways that are driven by human needs and desires, such as in relation to agriculture, hunting or conservation. Here's an example from a scientific text:

> Without estimates of mortality and births that are unbiased, precise, and accurate (approximating the true values with high certainty), policies that promote the killing of wildlife will risk unsustainable mortality and raise the probability of a population crash. The current government of the state of Wisconsin risked that crash when it issued high wolf-hunting quotas and when it liberalized culling from 2012 to 2014, both done without presenting careful, transparent accounting of mortality and births. (Treves et al., 2017, p. 29)

These discourses reflect the human exemptionalism paradigm that see humans as managers of the natural world, whether for purposes of economic exploitation (e.g., agriculture) or conservation (many of the scientific texts that I found were about conservation). The mathematics is familiar, even comfortable, and provides an apparently clear portrait of the state of wolf populations, as well as how they may best be managed through "harvesting" a certain number of wolves.

The poem, *Wolfwatching*, is, among other things, an observation of a wolf in a zoo. The poem evokes a different kind of relation from that engendered by mathematical models; it highlights the individuality and dignity of the wolf and thus provides a contrast to the mathematically produced relation between humans and wolves. The poem highlights the being of wolves, as well as the relation between humans and wolves through the act of being in relation with an individual wolf. The poem evokes the lupinity of the wolf through describing its situation in captivity, its condition (getting old) and implies how the wolf might feel about its situation. For example, the wolf's eyes are described

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[4] Stanley, G. (2021, February 25). As Minnesota considers wolf hunt, Wisconsin hunters blow past quotas. *Star Tribune*. https://www.startribune.com/as-minnesota-considers-wolf-hunt-wisconsin-hunters-blow-past-quotas/600027577/?refresh=true

as “withered in / Under the white wool”, showing his age, as when “He yawns / Peevishly like an old man.”

Reading the poem, having read about the wolf cull, made me think about the lifeworld of each wolf, their relations with the other members of their pack, their place in the complex web of their ecosystem. It made me think about each of the 216 wolves that we killed in the cull and about the impact on their relatives; by impact, I mean the emotional effect of losing pack members, where each wolf is an individual. Set against the mathematically informed discourses of wolf management, the poem allowed me to see how mathematics obscures these relations, rendering wolves collective and anonymous, and also how this relation is obscured, due to the discourse of mathematics as a direct description of the world, rather than as a form of watching. The poem, then, highlights the connection between two beings (the wolf and the watcher) in a way that the mathematically informed texts did not. That is, a series of mathematical models or statistical information obscures the relation between humans and wolves, as well as the relations between individual wolves and the relations of wolves within their ecosystem. The question then arises as to whether mathematics can be implicated in discourses that keep such relations visible.

## Yasmine’s example: What did I do, this time?

For years, I have taught Ontario mathematics, both to students and to pre-service teachers. I have always tried to incorporate two things into my teaching: 1. my own cultural experiences and ideas; and 2. issues of social and ecological justices. I invite my students to do the same. I learnt to do this from the vast body of literature in critical mathematics education research endorsing both these points. I was relatively comfortable in my teaching (in my being of who I am as a teacher), until I was asked to teach a dominant form mathematics in a Canadian Indigenous context (Abtahi, 2022).

Obscured by my own comfortable teaching position, I started this teaching journey by doing the two things I used to do (see things 1 and 2, mentioned above). In a circle of teaching, interacting with my students and reflecting on my teaching, I sensed strongly that something was not right. Was it me, a Pers teaching Western mathematics to Canadian Indigenous colleagues? Was it the mathematics that I was teaching? The course was only 40 hours long but my internal tension expanded over months. This is where the philosophical standpoint of Rumi, written in the form of poetry helped me make sense of my tense internal feelings.

Rumi has a short poem that portrays a construct of life as beings-in-relation. He relates living to making circles using a drafting compass. To live—as effective as a circle that is drawn by compass—one needs to have a leg rooted in personal experiences and reflections and a leg that moves and makes relations with humans, plants or animals that one is in relation with. I called Rumi's portrayal of an individual's life “self-in-relations.” In his poem, Rumi is not telling me how to think or what is right or wrong. Instead, it leaves me to make sense of the imageries of life, the compass, its two legs, and drawing circles based on my own experiences. This free and figurative form of the poem evoked alternative stories, with which I was able to re-make sense of mathematics and its teaching, different from what I used to sense. In teaching Western mathematics in a Canadian Indigenous context, instead of thinking about mathematics as a set of knowledges listed in the Ontario curriculum, I started thinking about the kinds of relations mathematics and its teaching and learning put us into and how these relations affect us.

My open interpretation of the poem made me think about myself and my students as selves-in-relations. Each 'self' having two legs, one rooted and one moving. We were all rooted in our different kinds of literature, experiences, stories, worldviews, and knowing, making dynamic relations with each other and with mathematics. It is rather obvious that my students and I were making dynamic relations (teachers and students quite often do this). My emphasis here is that we were making dynamic relations with mathematics too.

Through the interpretive lens of Rumi's poetry, the mathematics I was comfortable teaching was becoming less and less comfortable. So were my methods of teaching (i.e., with the two things). Let's assume that I actually tried my best to incorporate my cultural understandings and examples into my teaching. Let's assume that my students did the same. It was mathematics that remained rather rigid. No matter how inclusive I tried to be, I couldn't stop noticing that I was still bringing the mathematical knowledge(s) that the Ontario curriculum deems to be the "right" kinds of knowledge(s) to acquire; examples include the teaching of fractions and square numbers and the teaching of time using clocks. The types of mathematical knowledges that I taught ignored my students (and also myself) as selves-in-relations. Neither the knowledges of mathematical concepts, nor the pedagogy of teaching and learning of these concepts was attuned to the two metaphorical legs of the students, rooted and moving in relations valued in Canadian Indigenous communities. It was through the understanding of these relations that I was able to express my tense feeling. I realised that my teaching of Ontario mathematics could have been harmful to my students, as it ignored their selves-in-relations.

Mathematics as a system of knowledge should help us notice things, reflect on them, and make decisions. This is similar to what Skovsmose (1994) refers to as the formatting power of mathematics. Nevertheless, the knowledge of mathematics alone did not help me notice or imagine the webs of relations my students—whose experiences and ways of living may be very different from mine—were in. The imaginary message of Rumi's poem did. I do not take this visibility lightly; as I argued elsewhere, if we cannot have such imagination, we are likely to act in ways that are harmful to these others' individual or communal well-being (Abtahi, 2022).

## Reflections

We reflect on these two examples to discuss what prompted us to draw on poetry in our writing, as well as how this poetry brought into relation different dimensions of mathematical and non-mathematical experience. In both cases, poetry led us to see and to feel relations with Others, that are missing in mathematics. In both examples, unless we imagine the others as selves-in-relation, our actions, no matter how mathematically correct, may be harmful to their individual or communal well-being. The language game of poetry, we suggest, helps to make salient this important idea.

Another important point that helped us reflect on the usefulness of the poetry is the difference between the more truth-based outcome of making sense of mathematics and the more relational-based outcome of making sense of poetry. Making sense of mathematics leads to a kind of truth, such as the human need to kill a calculated number of wolves yearly in order to 'manage' the population. Making sense of poetry leads to an understanding of the relations involved when we kill "mathematically" just one wolf, we are killing a mother caring for five wolf pups, or an alpha male who leads a pack. Or that what we might think of as the right mathematics to teach could be harmful to the students whose rooted legs and the legs in relations are ignored.

These examples lead us to propose that thinking with poetry alongside thinking with mathematics leads to a valuable dialogue of perspectives and ideas. We are not suggesting that mathematics should be turned into poetry or that poetry should be incorporated into mathematical thinking (although these might be interesting things to explore). Nor are we arguing that poetry could be useful for teaching mathematics or could lend itself to ways of analysing data collected in mathematics classrooms. Rather, we are suggesting that by reading the world with poetry alongside reading the world with mathematics can lead to a critical understanding of what mathematics is doing. Wolfwatching expresses something about wolves that mathematics cannot capture, thus revealing a blindspot. Rumi expresses something about humans' place in the cosmos that mathematics curricula do not generally include. In Rorty's terms, poetry offers alternative stories, and alternative ways of constructing stories, from those available in mathematics or mathematics education.

Our goal in this paper was not to provide a definitive method. We have shared some reflections of our own engagement with poetry and mathematics, setting them in relation to each other and noting the new awareness this provoked in us. We hope our examples prompt further reflections drawing on other examples. We also wonder what other language games could be brought into relation with mathematics and what awarenesses they might prompt.

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# Leonardo Da Vinci's methods of calculation of the area of a circle and their counterparts in modern mathematics textbooks

Viktor Freiman[1] and Alexei Volkov[2]

*Our paper discusses Leonardo da Vinci's methods of calculating the area of a circle which seem to be reflected in modern curricula (e.g. in Canada, NB) and teaching materials, online (e.g. GeoGebra applets) and printed (modern textbooks). Among the historical sources we analyze Da Vinci's original drawings (e.g. Codex Atlanticus), his explanations of the procedure and (in Latin), and its French (by Ravaisson-Mollen and Russian (by Zubov) translations. Didactical implications are briefly discussed in the conclusion.*

*Keywords: Area of a circle, rearrangement method, da Vinci's drawings, procedure for calculation, didactical implications*

## Introduction

In our paper we pursue our study of the "matching sectors method" (or "rearrangement method") used to justify the formula of the area of a circle. This method is often mentioned in school curricula and found in numerous textbooks as well as in teaching materials available online (Freiman & Volkov 2019; 2022). It appears plausible to conjecture that this method was originally used by professional mathematicians of Antiquity and the Middle Ages in the West and East,[3] and only later was borrowed by mathematics educators and placed into school textbooks. This transfer thus may be considered a case of "didactical transposition" from "knowledge to be used" to "knowledge to be taught and learned" (Chevallard, 1985 [1991]; Chevallard & Johsua 2007; Chevallard, Barquero et al., 2022;[4] Kang & Kilpatrick, 1992). The earliest cases of the "rearrangement method" that we were able to identify in the Western textbooks were produced in the first half of the 19th century (Lardner, 1835, 1840). It became very popular since then, and various versions of it can be found in numerous Western mathematics textbooks of the 20th century (Freiman & Volkov 2022).

What were the particular reasons for the use of this presumably archaic method for instruction in the 19th and especially in the 20th century? Was its use a direct continuation of some didactical traditions of the remote past or an attempt to improve teaching of a relatively difficult topic with the help of innovative didactical approaches combined with refurbished ancient methods? It remains equally unclear where from did the method of "matching sectors" come to the modern school textbooks and

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[3] We shall provide a detailed discussion of the similar methods found in Chinese and Japanese mathematical treatises in a forthcoming publication.

[4] See the chapter authored in this book by C. Winsløw for a plausible discussion on the introduction of the concepts of the length of circumference and of the area of a circle (Winsløw 2022, esp. see pp. 300-301).

why was it perceived as an especially “efficient” tool for teaching. Was it because Archimedes’ (287-212 BC) “exhaustion method”[5] involved a rather sophisticated reasoning and thus was considered difficult for beginners?[6] Was it because the method of “matching sectors” used in textbooks may have seemed more “didactically attractive” since it looked easy enough to be understood even by relatively young learners? This simplicity, along with visual and dynamic nature of rearrangements, most probably seemed to the modern educators to be the most efficient way to construct classroom activities, which would be “hands-on,” practical, investigative, and providing students with initial intuition that could lead to more complex mathematical concepts (see, for example, Menghini, 2015). However, while some of these methods indeed allowed a relatively simple visualization, some others involved concepts and ideas particularly difficult for young learners. The infinitesimal methods of calculation of the areas of a circle and its parts arguably belonged to the latter category. The search for appropriate didactical tools applicable in this case led some mathematics educators to the use of the “method of rearrangement of sectors” or, more precisely, of a group of methods based on the division of a circle into sectors followed by rearrangements of these sectors and investigation of the ensuing approximations. For instance, Johnson and Mowry (2016, p. 574) provide a typical interpretation of the method. The demonstration begins with a small number of sectors (say, eight) (See top left figure in Figure 1). Then the authors state that once the number of sectors increases “the rearranged figure becomes quite rectangular” (See bottom right figure in Figure 1). One can immediately notice that the upper and lower sides of the figure, which are constructed of 32 sectors are shown as *straight lines* while its left and right sides are *vertical*. Both representations are therefore mathematically *erroneous*. Moreover, the equality of the areas of the circle and of the rectangle remains unexplained. The words “quite rectangular” certainly do not suffice to prove that the area of the circle is equal *exactly* to the product of the radius and circumference. Apparently, much of the work dealing with these subtleties is left to the teacher.

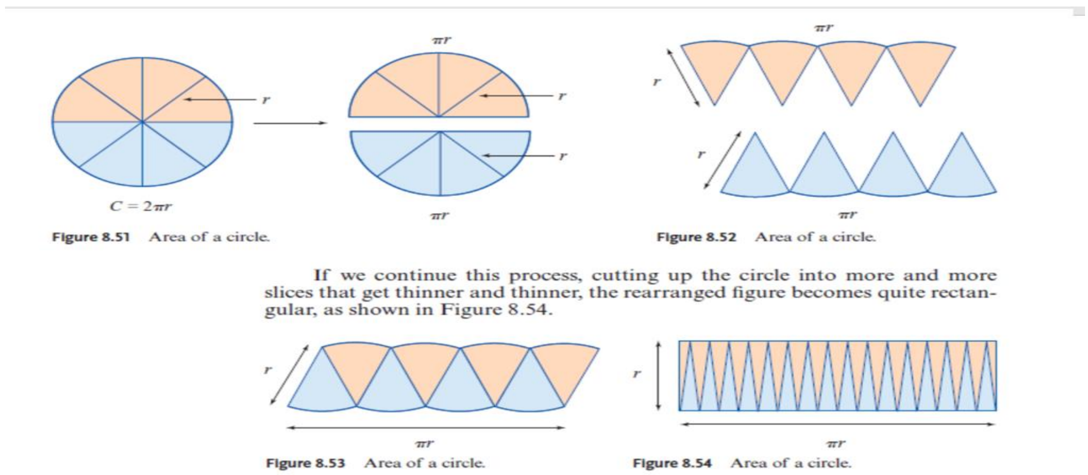


**Figure 1: Calculation of the area of a circle according to Johnson and Mowry (2016, p. 574)**

[5] Archimedes 1544, pp. 44-46; 1615, pp. 128-133; 1676, pp. 81-97; 1880, pp. 257-271.

[6] For a discussion of Archimedes’ method see, for instance, Dijksterhuis 1987 [1956] (esp. see pp. 222-240); Knorr 1976; 1986; 1989, pp. 375 ff.

These didactical considerations led us to the following conjecture about the concepts of the mathematics *educators* who designed these approaches. They arguably adopted an assumption concerning the growth of the child's understanding of geometrical concepts as mirroring the stages of the historical genesis of these concepts (Schubring, 1978, 1988; Simon, 1995; Mosvold, 2002). This conjecture most likely prompted a search of earlier sources of the method that presumably inspired the authors of the studied textbooks.

The preliminary results of our investigation can be summarized as follows. Several authors of the modern textbooks mentioned the works of Johannes Kepler (1571–1630) (Baron, 1969) and of the Japanese mathematician Satō Moshun 佐藤茂春 (whose name can also be read as Satō Shigeharu, dates of life unknown) (Smith and Mikami, 1914)[7] among others. Alexander Bogomolny's site refers to Petr Beckmann's (1924–1993) *A History of Pi* (1976 [1971]) who, in turn, mentioned Leonardo da Vinci's (1452-1519) method; the latter thus became the focus of the present paper. Indeed, page 518, *recto*, of the collection of Da Vinci's manuscripts titled *Codex Atlanticus* contains drawings showing the rearrangement of a circle divided into 16 sectors; the resulting shape is "close to rectangle" and is used by Leonardo to calculate of the area of the circle (Fig. 2). On the back side of the same page (p. 518 *verso*) one can find a more detailed drawing of the steps to be made to calculate the area of the sector of a circle; see Fig. 3. www.cut-the-knot.com refers to Petr Beckmann's (1924–1993) *A History of Pi* (1976/1971) which, in turn, mentions Leonardo da Vinci's (1452–1519) method. The latter thus became the focus of the present paper. Indeed, page 518, *recto*, of the collection of Da Vinci's manuscripts titled *Codex Atlanticus* contains drawings showing the rearrangement of a circle divided into 16 sectors; the resulting shape is "close to rectangle" and is used by Leonardo to calculate the area of the circle (Figure 2). On the back side of the same page (p. 518 *verso*) one can find a more detailed drawing of the steps to be made to calculate the area of a sector of a circle; see Figure 3.

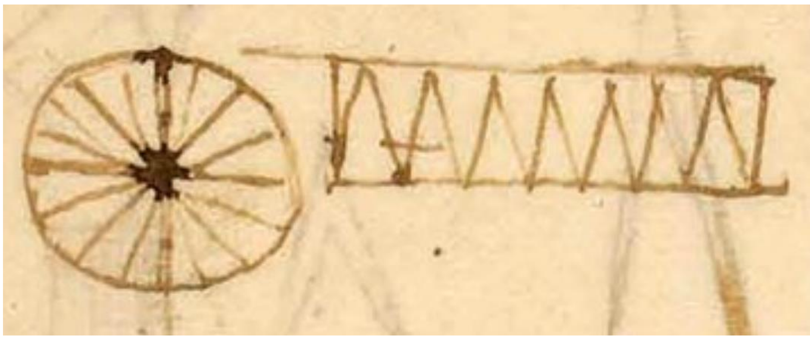

**Figure 2: A diagram from Leonardo da Vinci's *Codex Atlanticus*, p. 518 *recto*. The circle is subdivided into 16 sectors (da Vinci, 1513).**

[7] See Chapter 9 of the treatise of Satō's treatise titled *Kaisei tengen shinan* 改正天元指南 (Modified and Corrected "Compass of the [method] of Celestial Element") published in 1795. For information on Satō see Smith and Mikami 1914, pp. 86 ff.

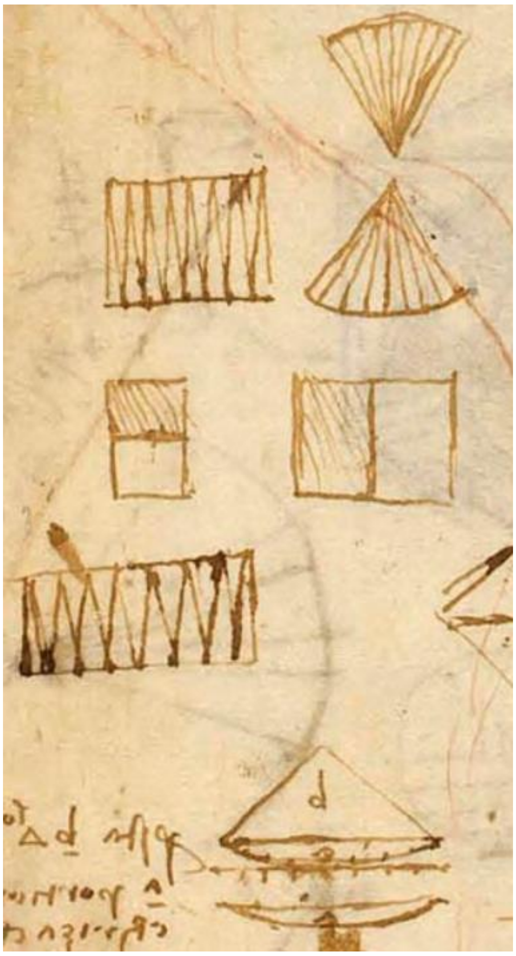

**Figure 3: Diagrams from Leonardo da Vinci's *Codex Atlanticus*, p. 518 *verso*. Calculation of the area of a sector. The sector is subdivided into eight smaller ones, then its identical copy is attached to it in order to form a rectangle (da Vinci, 1513).**

It is still unclear what were the sources of Da Vinci's drawings and of his short notes related to the area of a circle. However, these drawings suggest that one should not take for granted the claim made by Beckmann:

> "[Da Vinci] did not have much of a mathematical education, and in any case, he could use little else, for Europe in his day, debilitated by more than a millennium of Roman Empire and Roman Church, was on a mathematical level close to that achieved in ancient Mesopotamia. It seems probable, then, that this was the way in which ancient peoples found the area of the circle" (Beckmann, 1971/1976, p. 19).

Indeed, the complexity of Da Vinci's approach to acquisition of scientific knowledge deserves a separate study. Here we can only briefly mention Henri Lemonnier's (1842–1936) remark made in his analysis of Pierre Maurice Marie Duhem's (1861–1916) study of Da Vinci's heritage:

> Les historiens de Léonard, lorsqu'ils ont étudié le savant et le penseur, ont généralement cherché et trouvé en lui « l'autodidacte par excellence », et ils voyaient là une gloire de plus, celle de l'homme dont le génie a tout deviné sans avoir eu de précurseur[8] (Lemonnier, 1917, p. 27).

On the contrary, Duhem tended to highlight Da Vinci's capacity to read and assimilate what he was reading. On the same page, Lemonnier cites Duhem as follows :

> Non seulement les notes manuscrites de Léonard montrent qu'il avait beaucoup lu, mais elles témoignent de l'admirable puissance avec laquelle il s'assimilait tout ce qu'il lisait…Il est telle proposition de mécanique, d'hydraulique, de géologie, dont nous avons pu avec certitude retrouver la source, qui n'est assurément qu'un souvenir de lecture, et dont il est facile de relever quatre, cinq, six énoncés légèrement différents les uns des autres... [9,10]

At this point, the connection between Da Vinci and Luca Pacioli (1445–1519) needs to be discussed. Pisano (2016), for example, points out that being "above all a clever and very creative polymath but certainly not a mathematician like, for example, Luca Pacioli" (Pisano, 2016, p. 104), Da Vinci, while focusing on practical areas in which he operated as "a technician", he felt a need for "mathematical and geometrical abstraction" (p. 104). In this respect, Da Vinci's friendship with Pacioli extended the scope of influence the latter could have on his relation to mathematics. However, Pisano (2016) argues, "Leonardo improved his knowledge, particularly of Geometry, but despite this, his achievements remain immature as compared with the deep mathematical ideas of Pacioli (p. 104)." In turn, Pacioli's work might have shown an indication of an Archimedian influence. For instance, Høyrup (2022) states that Pacioli in his *Summa de arithmetica* (1494) "refers in the dedication to Duke Guidobaldo to 'the great Syracusan geometer Archimedes' who with 'his machines and mechanical inventions kept Syracuse safe for long'" (p. 185). Moreover, on the same page Høyrup also cites Marshall Clagett's (1916–2005) work *Archimedes in the Middle Ages* (Clagett, 1978) to stress that Da Vinci eventually drew on "medieval Archimedean mathematics."

Høyrup's quotation from one of the translations of Da Vinci's works is of particular interest for the purpose of our analysis of Leonardo's study of the area of a circle. It shows Da Vinci's understanding that Archimedes "never squared any figure with curved sides" but "only squared the circle minus the smallest portion that the intellect can conceive, that is the smallest point visible" (Høyrup 2022, p. 185). In order to study Da Vinci's representation of the method, we used his original manuscripts (Da Vinci, 1888, 1890) published by Ch. Ravaisson-Mollien (1848–1919) and their translations by V. P. Zubov (В.П. Зубов, 1900–1963) (Zubov, 1935, 1955).

## Leonardo da Vinci and his method of calculation of the area of a circle

In his second volume of his works on Leonardo da Vinci P. Duhem devoted a short chapter titled "L'infiniment grand et l'infiniment petit dans les notes de Léonard de Vinci" (1909, pp. 49–53) to

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[8] English translation: The historians of Leonardo, when they have studied the scientist and the thinker, have generally sought and found in him "the autodidact par excellence", and they saw in this one more glory, that of the man whose genius has everything guessed without having had a precursor.

[9] In a footnote Lemonnier specifies that he refers to pp. 340-341 of Duhem 1909.

[10] English translation: Not only do Leonardo's handwritten notes show that he had read a great deal, but they testify to the admirable power with which he assimilated everything he read... There is such a proposal in mechanics, hydraulics, geology, whose source we have been able to find with certainty, which is certainly only a memory of reading, and from which it is easy to identify four, five, six statements slightly different from each other...

Leonardo's concepts of infinitely small and infinitely large entities. Duhem quoted Leonardo's claim that "La Géométrie est infinie parce que toute quantité continue est divisible à l'infini dans l'un et l'autre sens[11] (p. 50). Duhem borrowed this quote from Manuscript M of the Bibliothèque de [12] and provided his own reconstruction of Leonardo's drawing that was supposed to prove (or at least to illustrate) Leonardo's claim. Surprisingly, Duhem does not mention here Leonardo's fragment and diagrams directly related to the calculation of the area of a sector and a circle [13] The text of this fragment of Leonardo reads as follows (we keep the original spelling and punctuation):

> QUADRATURA DEL SETTORE LV — Presta il triangolo a b c[14] alla portione b c d ediuidilo in settori come sidimostra ne la seconda fighura g h i k di poi separaliangholi dessi sectori lun dallaltro in modo che tal sia losspatio interposto infra essi angholi quale he lebase spia nate dessi sectori[.] Dipoi pre sta alli settori della terza fighura r s t v altrettanti settori ci o a e altrettanta valuta earai fac to vn quadrilatero chee n m o p[.] Fatto ilquadrilatero della quarta fighura leuanelameta enraile vato *lo sotto sector* li settori presstati erestera vna quanti ta equale alla seconda fighura g h i k[.] *Dipo* laqual restera quadrata[.] — Oltre addi quessto leverai tanto des so quadrato chessia equivalente al triangolo della prima fighura a b c eressteratti la portione dun ci erchio quadrata cioe la portione b c d *laqu* della quale illato cur vo sidirizzo cholmoto fatto sopra laretta *h* e d f ecquesta bella sola e vera reghola da dare la quadratura dongni portion di cier chio minor del semi circhulo della quale nulla scientia vale senon chol prestare e chol moto predetto disopra ec.[15]

The French translation of this excerpt provided by Ravaisson-Mollien reads as follows:

> Quadrature du secteur 1v. Prête le triangle a b e à la portion b c d, et divise-le en secteurs, comme on le démontre dans la seconde figure g h i k; ensuite, sépare les angles des secteurs les uns des autres, de façon que l'espace interposé entre ces angles soit tel que sont les bases aplanies de ces secteurs. Ensuite, prête aux secteurs de la 3ème figure r s t v autant de secteurs, c'est-à-dire leur équivalence, et tu auras fait un quadrilatère qui est n m o p. Le quadrilatère de la 4ème figure étant fait, enlèves-en la moitié, et tu auras enlevé *le secteur de dessous* les secteurs prêtés; il restera une quantité égale à la 2ème figure g h i k […], qui restera carrée. Outre cela, tu enlèveras de ce carré ce qu'il faut pour qu'il soit équivalent au triangle de la première figure a b c, et il te restera la portion d'un cercle rendue carrée, c'est-à-dire la portion b c d *laq*, dont le côté courbe s'est dressé avec le mouvement fait sur la droite […] e d f. C'est là la seule et vraie règle pour donner la quadrature de chaque portion de cercle plus petite que le demi-cercle, pour laquelle aucune science ne vaut si ce n'est par prêt et par le mouvement dit ci-dessus, etc.

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[11] English translation: Geometry is infinite because any continuous quantity is infinitely divisible in either direction

[12] Da Vinci 1890, Ms M, fol. 18r.

[13] Da Vinci 1888, Ms E, folios 24v-26v. Note that on the first page of this manuscript Leonardo already gives his verbal formula for the area of a circle: "Ilcierchio he equale avnquadri latero fatto della meta deldiamitro ditalcierchio multiplichato nella meta della circhunferentia delme desimo cierchio" (we keep Ravaisson-Mollien's transcription of the original document). The latter author provides his French translation on the same page; it reads as follows: "Le cercle est égal à un quadrilatère fait de la moitié du diamètre de ce cercle, multiplié par la moitié de la circonférence du même cercle," that is, "The circle is equal to a quadrilateral made of a half of the diameter of this circle, multiplied by a half of the circumference of the same circle."

[14] Transcription of Ravaisson-Mollien reads "a b e"; see below.

[15] Da Vinci 1888, vol. 3, Ms E, folio 25r.

Indeed, this note is located within a series of folios devoted to manipulations with the circles and spheres and accompanied by comments of Da Vinci. More specifically, Folio 24 (*verso*) deals with the surface of a sphere, which is first divided into eight equal parts, and then these parts go through a series of transformations (such as ‘flattening’ and ‘rearrangement’) to complete a rectangular figure (Figs. 4–8). While the process described by Da Vinci is unclear and would require a separate study, one can clearly see in these pictures that each of two shapes representing a quarter of a circle is cut into 8 smaller equal sectors and then combined with the other eight to produce a figure resembling a “rectangle” in the same way that can be found in today’s school textbooks and online resources. The above-mentioned representation of a quadrature of a sector is found in Folio 25 (*recto*) in a discussion devoted to the quadrature of a segment of a circle (Figure 9). V.P. Zubov (1955, p. 72) cites the *Atlantic Codex* (p. 98v) in which Leonardo includes methods allowing to conduct quadratures of surfaces delimited by curves. Interestingly, Da Vinci mentions that the square is the “end of all transformations” of such surfaces.

**Figure 4: Figure from Da Vinci 1888, Manuscript E, Folio. 24 verso**

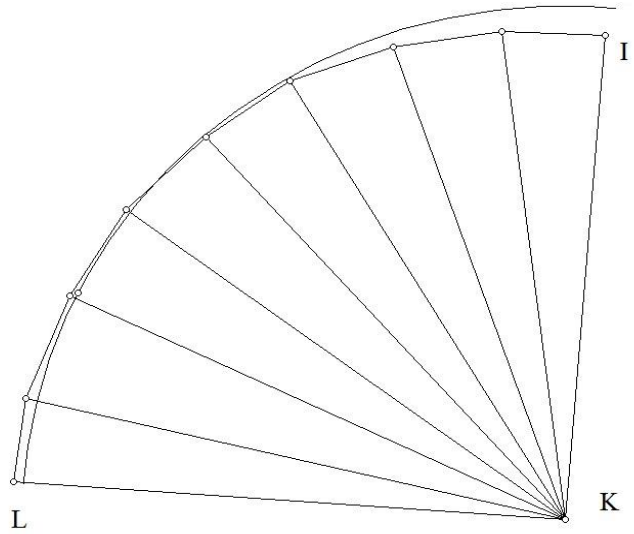


**Figure 5: Upper part of the figure from Da Vinci 1888, Ms E, folio 24 verso (reconstructed by the authors)**

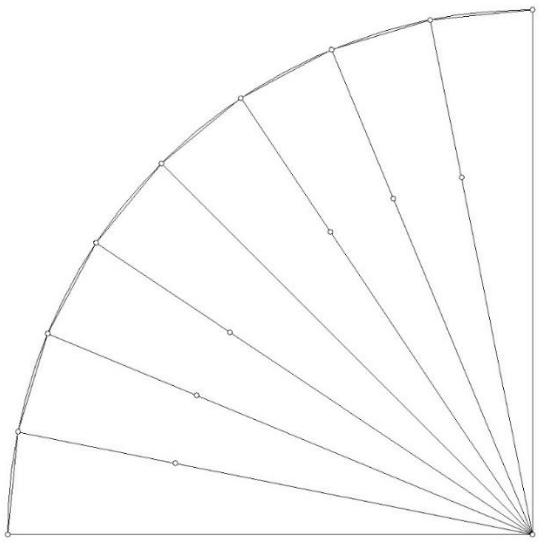

**Figure 6: Upper part of the figure from Da Vinci 1888, Manuscript E, folio 24 verso (reconstructed by the authors)**

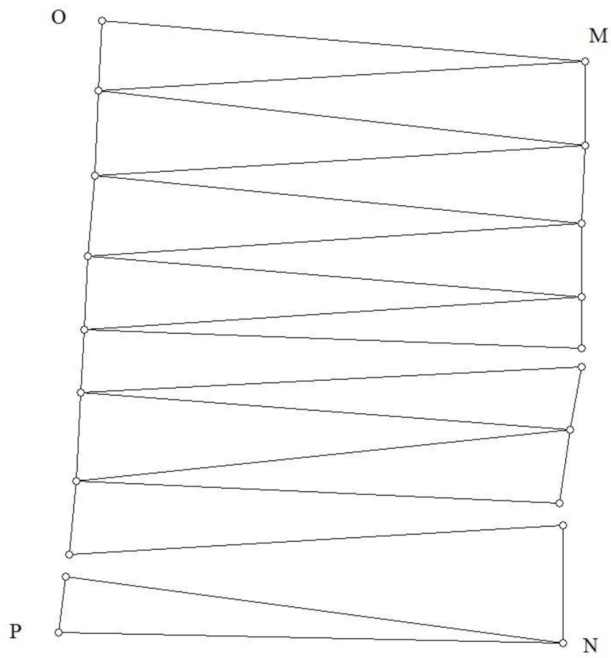


**Figure: 7: The lower part of the figure from Da Vinci 1888, Manuscript E, folio 24 verso (redrawn by the authors)**

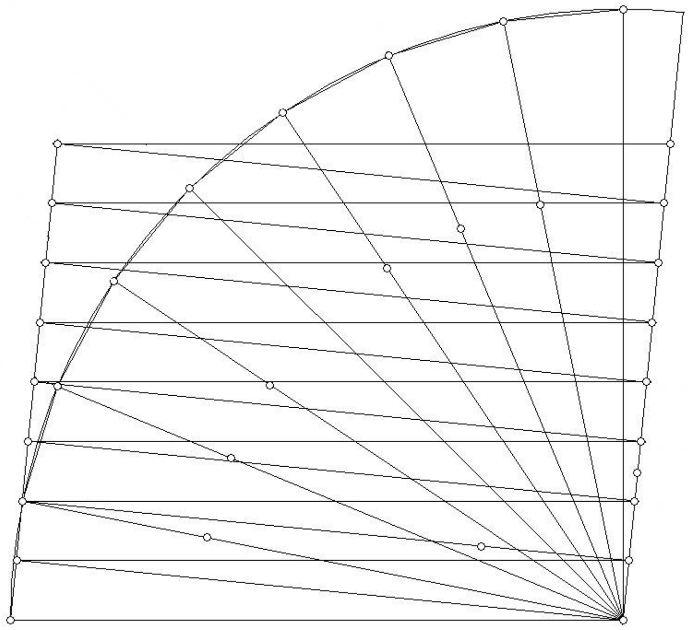

**Figure 8: Lower part of the figure from Da Vinci 1888, Manuscript E, folio 24 verso combined with the figure in the upper part (reconstructed by the authors)**

Our reconstruction of the method described in Da Vinci's note is based on the four sketches presented in Figure 9 where we see, in the first sketch, a segment of a circle to which the triangle *abc* is added in order to produce a sector —here Leonardo only briefly suggests "add the triangle". The second sketch shows the sector (seen as a segment combined with a triangle). which should be divided into smaller sectors (*ghik*) before one separates the angles of the sectors from each other in "such a way that the space between the vertices of these angles becomes equal to flattened bases of the sectors." The next step (shown in the third figure from the top right in Figure 9) consists of adding to the sectors of the third figure *rstv* the same number of identical sectors—and therefore having the area equal to their area—to complete the quadrilateral *nmop*. Once the quadrilateral is completed and its area is calculated, one should remove one half of it (that means that the added sectors are now removed) to get an area equal to the area of the sector (which thus will be "squared"). Finally, when one removes the part equal to the triangle *abc*, the remaining part will be a "squared segment of a circle the curved side of which has been straightened by the movement on the line *edf*". According to his final note, Da Vinci considered this method as the "only correct rule to give a quadrature of a part of a circle which is less than its half" (Figure 10) (We use quotes translated from the above French version (Da Vinci, 1888).

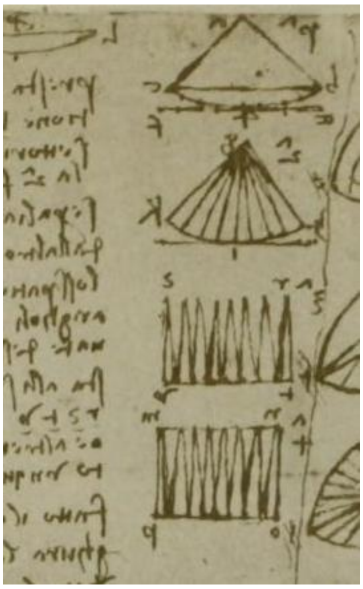

**Figure 9: Figures from Da Vinci 1888, Manuscript E, folio 25 recto (upper part)**

Our English rendering of this text is based on the translations of Ch. Ravaisson-Mollien and V.P. Zubov (1935, 1955). It reads as follows:

> Quadrature of the sector *lv*. Add the triangle *abc*[16] to the part *bcd* and divide it into sectors, as is shown in the second figure *ghik*; then separate the angles of the sectors from each other in such a way that the space between the vertices of these angles become equal to flattened bases of the sectors. After that add to the sectors of the third figure *rstv* the same number of sectors, that is, the area equivalent to their [area], and you will complete a quadrilateral, which is *nmop*. When the quadrilateral of the 4th figure is completed, remove one half of it, and you will remove the added sectors; the remaining part will be equal to the second figure *ghik,* which remains square. Then you remove from this square what is needed to make it equal to the triangle of the first figure *abc*, and what remains to you will be the portion of the circle made square, that is, the part *bcd*, of which the curved side was placed with the movement that [you] made on the line *edf*. This is the only and correct rule to give the quadrature of each

[16] In his translation Ravaisson-Mollien writes "abe" instead of "abc".

part of a circle, which is smaller than half-circle, for which no science is worth if it is not [done] by borrowing and by the movement discussed above, etc.

Then comes Folio 25 (*recto*) discussing a quadrature of a circle (actually, a fourth of it). Da Vinci's figures shown in Figure 9 include a quarter of a circle (the sector in the upper part of Figure 9) subdivided into two parts, a triangle ABC (mentioned in Leonardo's words 'add the triangle') and the segment BCD shown in Figure 10. The last folio of the series, 26r, provides yet another example of a quadrature of a curved surface, where again we find the same method based on division of a sector into smaller sectors, followed by a subsequent transformation into a rectangular shape (Figures 11–14).

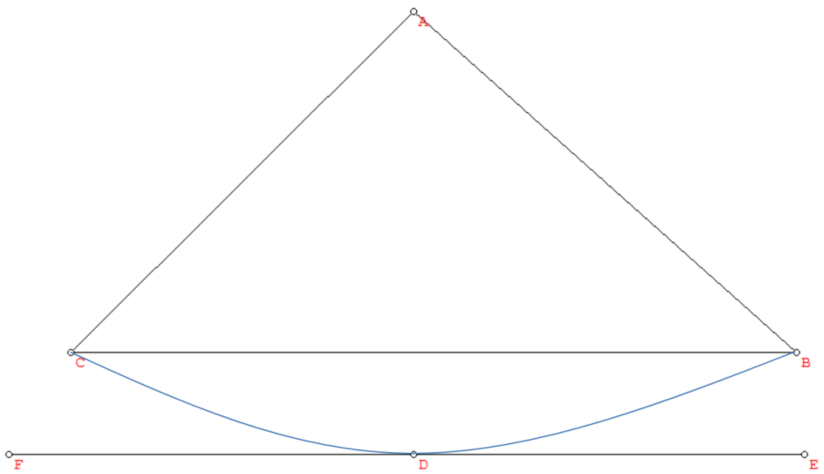


**Figure 10: Figure shown in the upper part of Figure 9 redrawn by the authors**

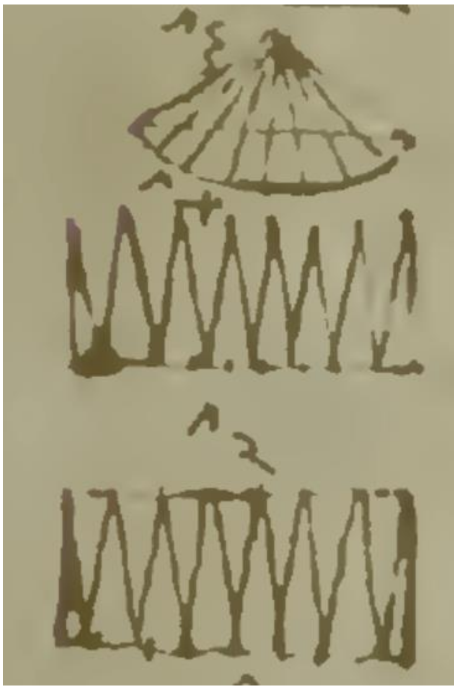

**Figure 11: Figures from Da Vinci 1888, Manuscript E, folio 26 recto (lower part)**

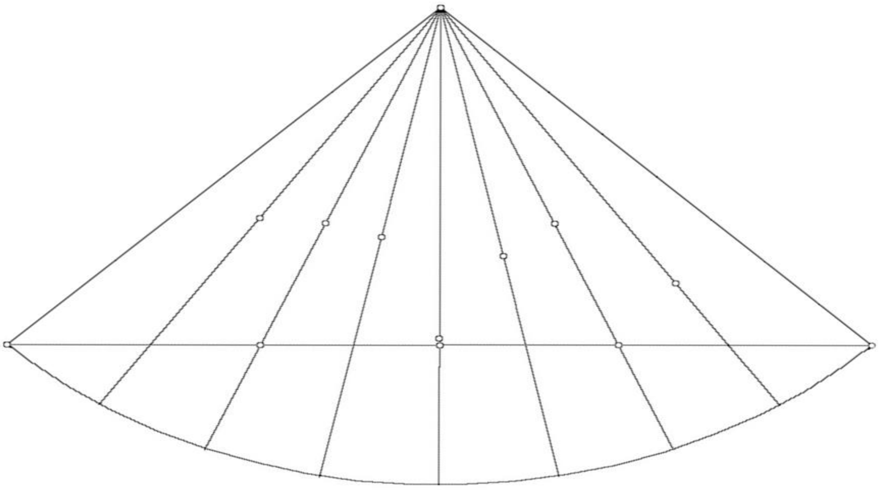

**Figure 12: Upper part of the diagram shown in Figure 11 (reconstructed by the authors)**

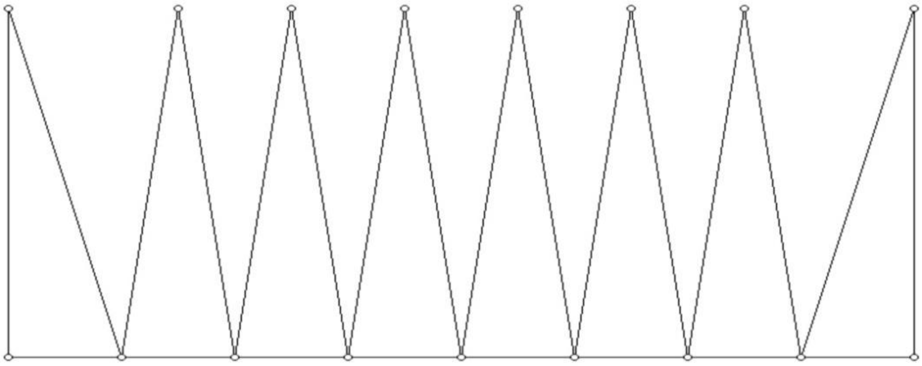

**Figure 13: Middle part of the diagram shown in Figure 11 (reconstructed by the authors)**

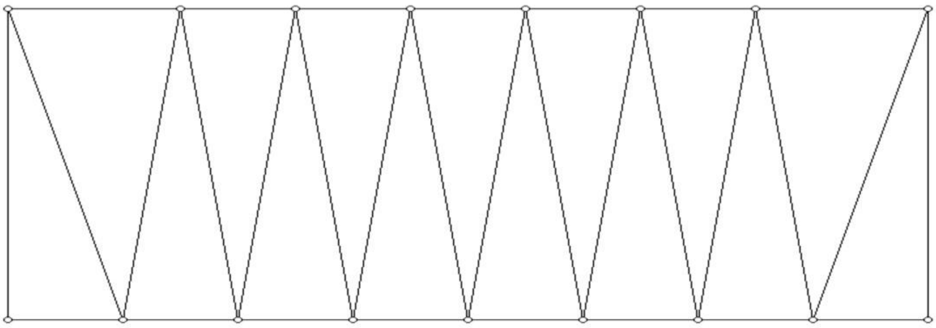

**Figure 14: Lower part of the diagram shown in Figure 11 (reconstructed by the authors)**

On the back side of the same *folio* (26v), Da Vinci provides a set of rules that can be used to calculate the area of a circle; these rules read as follows:

> De quadratura circulu
>
> Moitié contre moitié. — Entier — Multiplie la moitié de la circonférence du cercle par la moitié de son diamètre, et le résultat sera cette quadrature de cercle. Autre manière : Entier contre quart. — Multiplie toute la circonférence d'un cercle contre le quart de son diamètre, et tu auras la quadrature de ce cercle. Autre manière : Quart contre entier. — Multiplie le quart de la circonférence d'un cercle contre tout son diamètre, et ce qui en résulte sera la quadrature de ce cercle. Autre manière : Tout contre tout. — Multiplie toute la circonférence avec tout son diamètre, et du résultat ôte les [3/4?], \\ le reste sera ce que tu demandais. Autre manière : Moitié contre le tout. — Multiplie la moitié de la circonférence avec tout le diamètre, ou la moitié du diamètre avec toute la circonférence, et du résultat enlève la moitié; le reste sera un carré égal audit cercle. Tout contre le quart. —Multiplie le quart du diamètre d'un cercle contre toute la circonférence, et le résultat sera quadrature de son cercle.

Our English translation of this excerpt reads:

> On quadrature of circle
>
> Half against half. — Entire [figure] — Multiply half of the circumference of the circle by half of its diameter, and the result will be this quadrature of the circle. Another way: Whole versus quarter. — Multiply the entire circumference of a circle by a quarter of its diameter, and you will have the quadrature of this circle. Another way: Quarter against whole. — Multiply a quarter of the circumference of a circle by its entire diameter, and the result will be the squaring of that circle. Another way: Entire [one multiplied] by entire [one]. — Multiply the entire circumference by its entire diameter, and from the result take away its [three fourth?], \\ the rest will be what you asked for. Another way: Half against the whole. — Multiply the half of the circumference by the whole diameter, or half of the diameter by the whole circumference, and from the result subtract one half; the remainder will be a square equal to said circle. All against the quarter. —Multiply a quarter of the diameter of a circle by the entire circumference, and the result will be the quadrature of its circle.

## Discussion

In our paper, we conducted an analysis of the process of calculation of the area of a circle as illustrated in works of Leonardo da Vinci. Our choice of sources was based on references to Leonardo Da Vinci's works in modern literature, such as M. Baron's (1969) book on the history of calculus and P. Beckmann's *History of Pi* (Beckmann, 1971/1976), as well as in online resources (e.g., the *Cut-the-Knot* educational website created by Alexander Bogomolny, 1948–2018).[17] Our investigation was prompted by an analysis of the methods found in present-day textbooks whose authors apparently were focused on the didactical aspects of the infinitesimal methods based on intuitive, dynamic, and visual representations. It should be emphasized that the context of the work conducted by mathematicians of the past was *of a different nature.* They arguably focused on such problems as, for instance, the quadrature of the circle (Nicholas of Cusa, 1401–1464) or calculation of the volumes of the wine barrels (Johannes Kepler, 1571–1630) without being particularly concerned with didactical aspects of their methods. The need of calculation of the area of a circle arose from the contexts often dealing with issues related to solutions of more complex problems. As we showed in our earlier

[17] See the website http://cut-the-knot.org/.

publication (Freiman & Volkov, 2022), when one deals with textbooks specifically designed for learning, such as Alexis Claude Clairaut's (1713–1765) *Elémens* de *Géometrie* of 1741 (reprinted in 1830)[18] or geometry textbooks of the 19th and 20th centuries, it is rather obvious that the task of calculation of the area of a circle became essentially a *didactical* issue related to the problem of introduction of the infinitesimal methods and, in particular, of the concept of infinitely small entities. While it can be argued that the division of a circle into a number of equal sectors and their further rearrangement thus producing "teeth-like" diagrams in school textbooks was done mainly for the sake of visualization, Da Vinci's drawings seem to have been directly related to his concepts of infinitely small and infinitely large entities.

Interestingly enough, in the case of the area of the circle the epistemic issue of dealing with infinity interacts with the demand for visualization (in particular, with the need of using diagrams in textbooks) as well as with the use of induction (based on intuitive approach to infinitely small and large entities) and of approximations (often dictated by *practical* and not only by *theoretical* considerations). For instance, when one looks into Da Vinci's notes, a focus on drawings is apparent. For example, in Figure 3, two sectors of the circle are divided into smaller sectors and then appears a shape composed of these small sectors now looking like triangles. Then come two entirely abstract representations of a rectangle divided into two halves, one of which is shaded. Then again, a rectangle composed of "triangulated sectors" is shown followed by a very different drawing.We see a segment of a circle with an added triangle thus transforming it into a sector. What is even more interesting, there is a straight line, which seems to be equal to the arc of this sector.The arc of the sector is subdivided into equal parts which, when the arc of the sector is straightened, become equal segments of a straight line.

Using Ravaisson-Mollien's and Zubov's transcriptions and translations, we analyzed Leonardo's notes from another manuscript (Manuscript E), which has similar drawings. We found there two different interpretations of the method of "transforming a circumference into a piece of straight line" (with the length equal to the circumference). One of these interpretations was related to the category of methods based on the idea of "unfolding a circle" with sectors becoming a "teeth-like" rows of matching triangles. Methods of this category can be found in ancient and modern sources, such as, for example, in the diagram constructed by Casselman (2012) or in the one found in Beckmann (1971/1976). The other was based on the idea of "cutting out the sectors of a circle" (the sectors thus had to be moved and rearranged), when even the notion of "being equal" seemed to be sacrificed to the goal of making a new arrangement of parts fitting into a rectangular shape, see, for example, Lardner (1835, p. 114), Lardner (1840, p. 101), and Willis (1922). The latter approach can also be observed in Da Vinci's construction that reflected his vision of dynamically transforming shapes which, he claimed, would have an area "equal" to the area of a rectilinear figure. Hence, in connection to our didactical focus, we notice in Da Vinci's works certain ideas looking similar to modern didactical views of geometry as intuitive, visual, and dynamic discipline that can be studied with the help of visual representations and manipulations (vizualizing, cutting, transforming, and re-shaping) thus expecting to help learners to gain deeper understanding of the concept of the area of rectilinear

[18] On Clairaut's didactical ideas see, for example, Sander 1982 and Glaeser 1983.

and curvilinear figures. But one can also question the limitations of these methods when it comes to dealing with more abstract concepts of infinitesimal processes.

We argue that the division of a circle into a number of equal sectors and their further rearrangement thus producing “teeth-like” diagrams in school textbooks was done mainly for the sake of visualization; however, a number of minor differences between variants of this procedure can be identified. Furthermore, there is another aspect that deserves a deeper reflection as far as the *visual* part of the process is concerned. It is related to the *number* of sectors into which the circle was supposed to be dissected. More specifically, there existed two patterns that can be identified, the first one was based on a regular hexagon inscribed in the circle and then having its sides doubled thus generating the series of regular polygons with 12, 24, 48,…sides, while the second one featured regular polygons with the numbers of sides equal to powers of 2 (4, 8, 16…) and beginning with an inscribed square. Both visual representations were used to convince the reader (i.e., the learner) that the area of a given circle can indeed be calculated *via* an approximation by using polygons.

While the idea of dividing the circle into sectors is present in all models, there are differences in terms of their number as well as in their pictorial representations. For instance, some models provide illustration of the whole circle divided into equal sectors (Earl, 1894) while some others present a division of a half of a circle (Palmer, 1919), or show only a part of the sectors (Willis, 1922). There are also some less straightforward representations. For example, the model suggested in Henrici and Treutlein (1897) shows division of one half of a circle into three sectors and the other half into six sectors. When representing the result of a rearrangement, some models show the final result of only one transformation (Earl, 1894; Hall & Stevens, 1921; Palmer, 1919), or even leave the “matching” incomplete (like in Lardner, 1835, 1840). In his books, Lardner shows two rows of “teeth” close to each other but are not stuck together. Other authors prefer to show two consecutive transformations (like Henrici and Treutlein’s (1897) model with 3 and 6 pieces; or Willis’s (1922) one showing a part of sectors put together next to a completed rectangle). This variety of the representations of one and the same model might have had an impact on its use (in terms of instrumentation and of its didactical efficiency), which also deserves further investigation.

The division of the circle was followed by a *rearrangement part* arguably based on the assumption that the operations of division, decomposition, and re-composition of a given geometrical figure do not change its area. Several models that we discussed above reflect this dynamic process. At some point, the circle itself was treated (or should one say “defined”?) as a polygon *with an infinite number of sides*. This treatment immediately led to an even more complex (and complicated) issue, which remains mathematically and didactically challenging until now: How to deal with infinity? Some sources that we analyzed provided an explanation employing the concept of a “sufficiently large” number of sectors and claiming that the shape that could be constructed using the inscribed rectilinear figures “looks like” a triangle, or a rectangle, or a parallelogram. Others added the word “approximately” to the description of this visualization. Finally, a number of authors introduced some dynamic aspects into the process, explicitly or implicitly pointing at a possibility of increasing the number of sectors and, consequentially, making it intuitively clear that the area of the rearranged shape of the circle becomes closer and closer to that of a triangle, or a rectangle, or a parallelogram for which it was known how to calculate their areas. This dynamic process allowed for further manipulations with the formulas for the area of a triangle (or rectangle, or parallelogram) and, finally, for advancing conjectures concerning the formula of the area of a circle (most often based upon the previously proven formula for the circumference).

## Conclusions

While considering geometry as a complex activity embracing various processes of interaction between practical knowledge (e.g., measuring) and its theoretical codification (e.g., the Euclidean deductive system), we are inclined to see in the use of the "rearrangement method" an attempt to introduce a two-fold procedure explaining to the learners *how* to calculate the area of a circle and, *at the same time*, offering a plausible reasoning strategy explaining *why* this formula works. While considering it as an "instrumental activity" (implying both signs and tools) from its very beginnings, Richard et al. (2019, p. 143) reflected on the complexity of distinguishing between mathematics as a science and mathematics as "mathematical thought" applied in the context of particular situations, tasks, or activities. Richard et al. (2019) referred to Kuzniak and Richard's (2014) definition of mathematical work as a "progressively constructed process of bridging the epistemological and the cognitive aspects in accordance with three yet intertwined genetic developments as the *semiotic, instrumental and discursive* geneses" to introduce their model of MWS ("Mathematical Working Space") allowing to "report on mathematical activity, potential or real, during problem solving or mathematical tasks" (Richard et al., 2019, p. 144).

From the historico-didactical perspective, our analysis allows for some preliminary observations. Firstly, while it appears to be difficult, or even impossible, to make direct connections between mathematicians' work prior to the 18th century and didactical innovations of later periods, some of the issues that mathematicians were dealing with certainly merit attentive look of modern educators. For instance, da Vinci's idea that the angles of the sectors can be separated from each other in such a way that the space between the vertices of these angles become equal to flattened bases of the sectors is resonating with that of transforming rearranged sectors into configurations where sectors are approximated with triangles, and, when put together, form rectilinear shapes (larger triangles, rectangles, or parallelograms). The idea of Nicholas of Cusa (Cusanus) that the area of the circle may be found by the same means as by that employed for any other polygon, that is, by dividing it up into a number (in this case, an infinite number) of triangles, was also fruitful in terms of the treatment of "teeth-like" representations in later (and even very recent) sources.[19] Johannes Kepler's work, besides following Cusanus' method of "indivisibles," provides an insight into the procedure that points at approximation as a way of approaching the circle by polygons with a large (or even infinite) number of sides.

Secondly, when exploring references to earlier works by Cusanus, da Vinci, and Kepler, we found that modern authors (e.g., Boyer (1959(, Baron (1969), and Beckmann (1971/1976) seem to add details, which were *not* found in the cited works. Indeed, *neither Cusanus, nor Kepler explicitly discussed* the "rearrangement" of sectors.

Thirdly, despite a seeming similarity of the ways in which the versions of the rearrangement method were described in the 19th and in the early 20th century sources (see brief discussion in Freiman & Volkov, 2022)), we noticed certain substantial *differences* in representations that need to be reflected

[19] For discussions of Cusanus' quadrature see Uebinger 1895, 1896, 1897; Wertz 2001; Nicolle 1996, 2001, 2020.

upon from the teaching and learning perspective (e.g., number of sectors into which the circle is divided and which seems to be connected to a shape of a polygon inscribed in a circle; the representation (definition) of a circle itself as a polygon with an infinite number of sides; type of the rearrangement employed; or even more complex, how should we deal with the infinity (approximation versus becoming equal), etc.). In this respect, going deeper into earlier sources (such as Da Vinci's writings) can provide rich material for a historico-genetic analysis of the methods involving the concepts of infinitely large and infinitely small entities that was the main goal of this paper, which was conducted within the framework of our ongoing investigation on the historical roots of modern didactical methods.

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# A new method for constructing Penrose-like tilings by using traditional Iranian patterns

Mohammad Hossein Eslampanah[1] and Payam Seraji[2]

*After a short survey on different ideas about the connections between Penrose tiling and some traditional Iranian patterns, which are based on decagon, a new method for constructing Penrose-like tilings is obtained. In this method one uses only four types of traditional Persian motifs. We will also prove that this tiling has some characteristics similar to Penrose tilings.*

*Keywords: Penrose tiling, Kond-e-dah gereh patterns, Quasicrystal, Non-periodic tilings.*

## Introduction

In 1974, Roger Penrose, a prominent British mathematician and physicist, introduced a new kind of tilings (Penrose, 1974) whose characteristics were different from the known tilings. They had "*global fivefold symmetry"* which means that there is a point that if we rotate the tilings in multiples of $\frac{2\pi}{5}$ around, the tilings would fit with the original one, and would be non-periodic, i.e., there is no vector that if we transform the tilings by that vector, the new tilings would fit with the original one. One famous example of this tiling, which only uses two tiles, is known as "*Kite*" and "*Dart*" (see Figure 1). Figure 2 is another example of this kind of tiling, which uses two tiles, known as "*fat rhombus*" and "*tin rhombus*" (for more information, see Penrose, 1974).

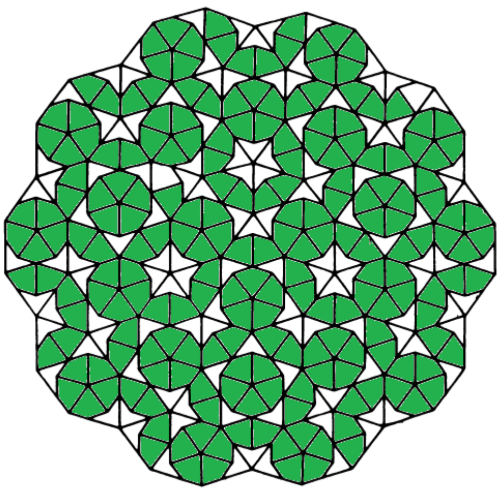

**Figure 1: Penrose's Kite and Dart tiling**

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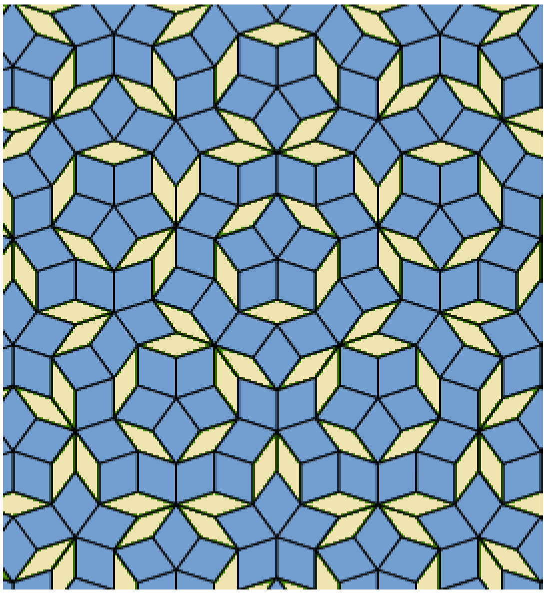

**Figure 2: Penrose thin and fat rhombus tiling**

A few years later, physicists discovered some structures, which are called "*quasicrystal.*" They found that relevant mathematical models for describing (some of) these structures are Penrose tilings (see Senechal, 2006).

In recent years, many scholars have suggested similarities and (possible) relations between Penrose tilings and some traditional Iranian-Islamic tilings, which are based on the regular decagon. Perhaps the most famous article on this subject was written by Peter Lu and Paul Steinhart (Lu & Steinhart, 2007), which even attracted much discourse in social media around the world. In this article, they introduce a set of five tiles ("*Gireh*" tiles). They show that these tiles have a self-similarity like property, i.e., we can construct a similar tile with a larger scale using some arrangements of tiles with smaller scales, a characteristic that holds also for Penrose tiles.

Historical evidence for using these tiles is their appearance in a pattern of Topkapi scroll[3] (Lu & Steinhart, 2007). Lu and Steinhart's main example is a tiling in Darb-e-Imam shrine (15 century, Isfahan, Iran). They claimed that:

> by the 15th century, the tessellation approach [using Gireh tiles] was combined with self-similar transformations to construct nearly perfect quasi-crystalline Penrose patterns, five centuries before their discovery in the West (Lu & Steinhart, 2007, p.1109).

Their claim has been criticized by some authors (for example Cromwell, 2009), but, whether all their claims are justified or not, it is clear that there are some relations between these two types of tilings and we can construct tilings with characteristics like Penrose tilings by using traditional Iranian-Islamic patterns (hereafter, for simplicity, we use the term "traditional patterns" instead of "traditional Iranian-Islamic patterns").

Another method for transforming a Penrose tiling to a tiling with traditional patterns is presented by Rigby (2006). He divides Penrose "Kite" and "Dart" by traditional patterns as it is shown in Figure 3. Then he shows that matching rules, which should be regarded for constructing Penrose tilings are

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[3] An architectural scroll from 15 century, now is held in Topkapi museum, Istanbul, Turkey.

compatible with this subdivision and the result will be a Penrose like tiling, which is constructed by using a set of five traditional motifs (Figure 4).

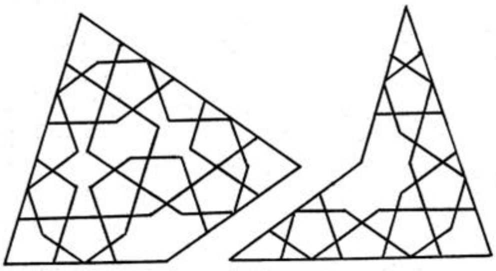

**Figure 3: Dividing Kite and Dart by Kond-e-Dah motifs**

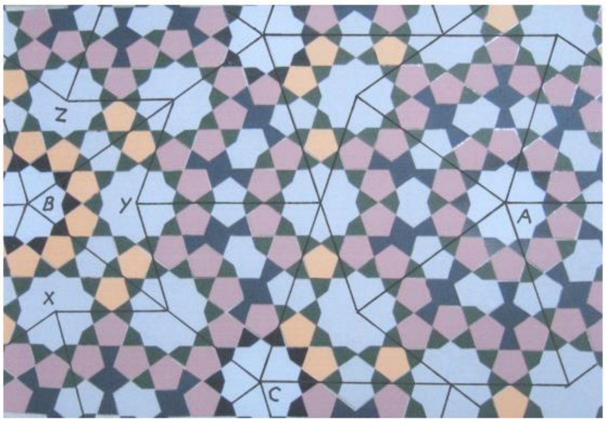


**Figure 4: Transformation of Penrose's "Kite" and "Dart" tiling to a tiling by traditional motifs (from Rigby, 2006)**

Subdivisions of Figure 6 can be seen in many old tilings in Iran, for example in Darb-e-Imam (Figure 5), or ChaharBagh school (Isfahan, 17 century, Figure 6).

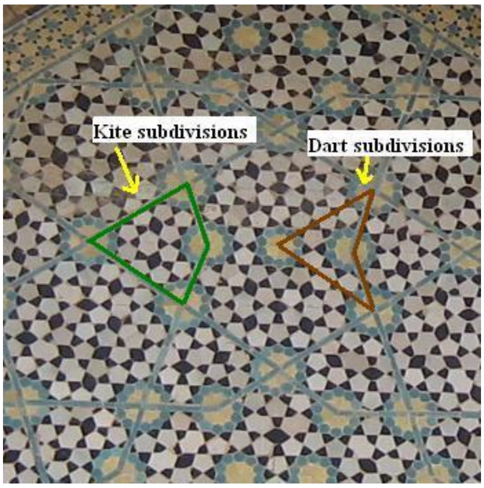


**Figure 5: Darbe-e-Imam shrine (photo by the authors)**

**Figure 6: Isfahan’s Chaharbagh school (photo by authors)**

These motifs are very common in Persian architecture and their traditional names are demonstrated in Figure 7. This Figure also shows how one can naturally draw these motifs by using a regular decagon (these are drawn by a master of traditional Persian architecture, Gh. Aghaebrahimiyan, (Eslampanah, 1996). Using these Figures, one can easily observe all the geometrical characteristics of these motifs. We only mention the followings:

1) “Torang” is exactly equal to Penrose’s “Kite”

2) Sides of all these shapes are equal or their proportions are one of the golden ratios (= $\frac{\sqrt{5}\pm 1}{2}$). and:

3) All angles are multiples of 36 degrees.

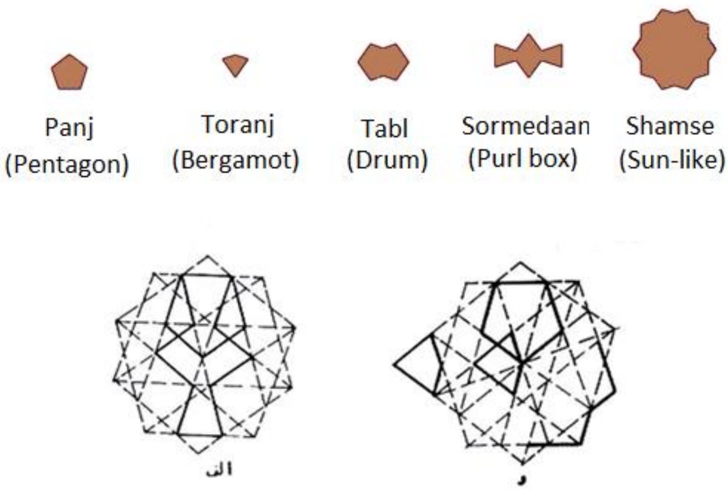


**Figure 7: Persian names for traditional motifs of Kon-e-Dah patterns and a method for drawing them using regular decagon (from Esslampanah, 1996)**

In what follows, we present a new method for transforming a Penrose tiling to a tiling which uses only four tiles of the above set (a part of this tiling can be seen in Figure 29). First step is to prove the following lemma:

**Lemma 1)** Any Penrose tiling by thin and fat rhombuses can be covered by two decagonal-shape tiles (which we call "type A" and "type B" decagons, Figures 8 and 9), if these tiles are allowed to have overlap in a thin rhombus or in an "Almond six"[4] (Figure 10).

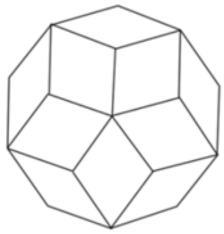

**Figure 8: Decagon type A**

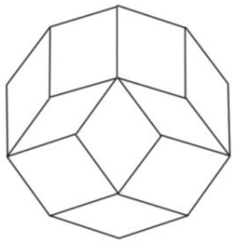

**Figure 9: Decagon type B**

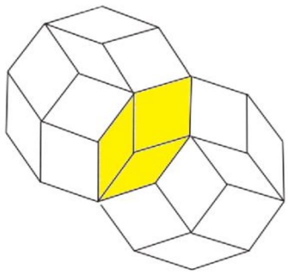

**Figure 10: Overlap in an Almond-six**

Proof: One method for describing local structure of Penrose tiling is considering the "*matching rules.*" Figure 11 shows one example which is proposed by Conway (Grunbaum & Shefard, 1990):

[4] Traditional name for this shape according to some architect masters

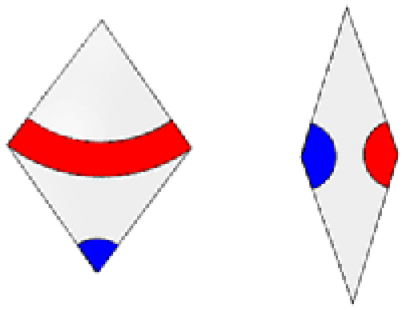

**Figure 11: Matching rules for thin and fat rhombuses**

Tiles should be adjacent in a way that red strips connect to each other and also blue strips connect to each other. By considering this rule and also using different ways in which we can fill 360 degrees with these tiles, it can be seen that there are only 8 possible ways for covering all 360 degrees around a point, by these two tiles (Figure 12).

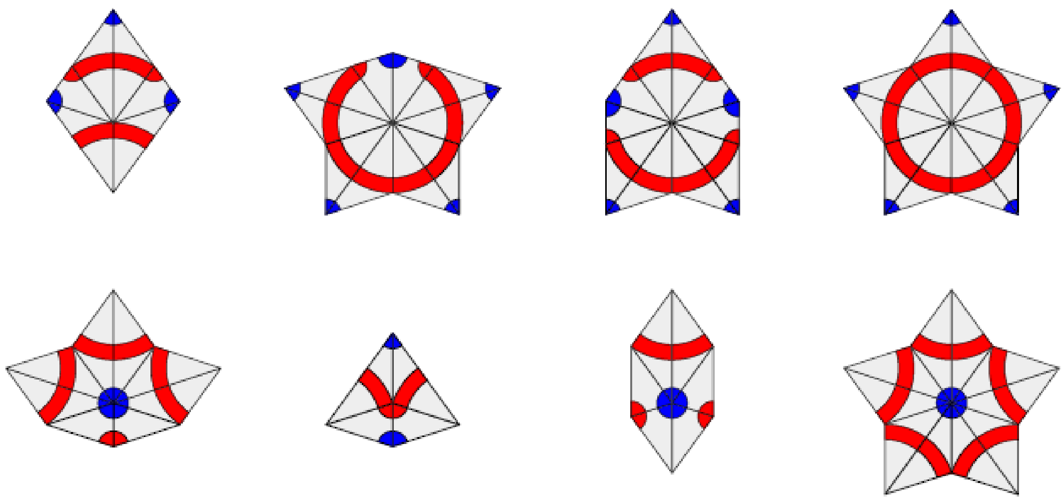

**Figure 12: Eight possible patterns around a point**

By considering each of these eight cases, one can see that using the matching rules, each central point falls inside, or on the boundaries of a decagon of type A or B. For example, in the following case (Figure 13), the only way for filling the points A,B,…,E is shown in Figure14, and the central point is inside a decagon of type A.

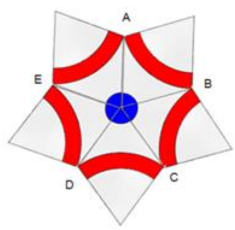


**Figure 13: Decagon of type A**

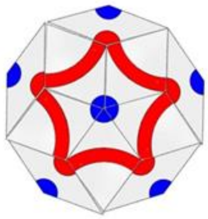

**Figure 14: Decagon of type B**

If two decagons have some overlap, then the path on the boundary of each decagon, which is inside another one, is a sequence of segments with equal length and each one makes an angle of 144 degrees with the next one (because inner angles of a decagon are 144 degrees). According to the structure of each type A and type B decagons, we have the following 3 possibilities (Figure 15).

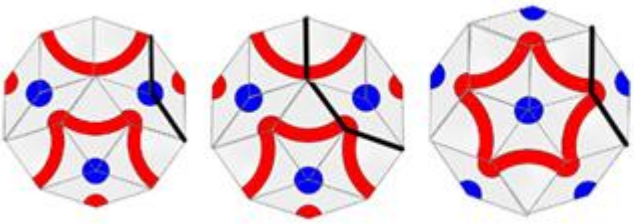

**Figure 15: Three types of intersection between decagons**

In each case, the overlapping space is a thin rhombus or Almond-six, which completes the proof.■

Now, we add to each of types A and type B decagons, some new lines, which fill these decagons with traditional patterns, in addition to some extra pieces on the sides (Figures 16 and 17).

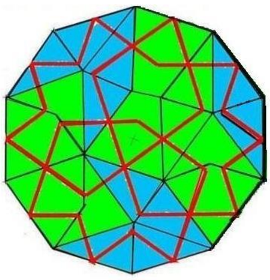

**Figure 16: Filling decagons of type A using traditional motives**

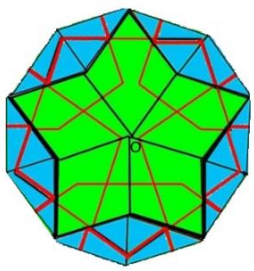

**Figure 17: Filling decagons of type A using traditional motives**

Justification of this filling is a simple exercise in geometry, and we leave this task to the reader. It can be easily checked that in all these three cases of overlapping spaces, traditional patterns fit on each other and there is no inconsistency between them.

**Lemma 2)** In all cases for which the edges of decagons of types A and B coincide, lines of traditional patterns join together and form a "Panj" (regular pentagon).

Proof: Two edges of a type A decagon cannot coincide in a Penrose tiling, because, as Figure 18 shows, it is impossible to fill around the point A in a way that matching rules work.

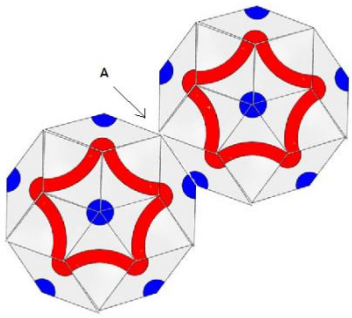


**Figure 18: First type of impossible matching**

By a similar reason, two decagons of types A and B cannot coincide in an edge. The only remaining case is that two decagons of type B have a common edge (Figure 19).

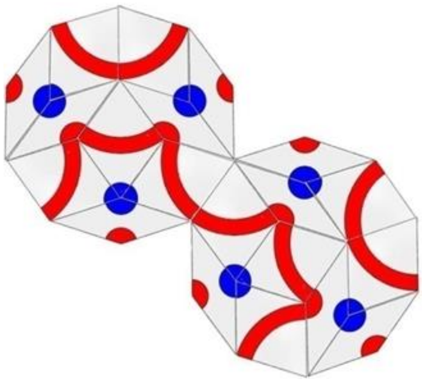

**Figure 19: Legal matching of two decagons**

It can easily be checked that in all possible ways for filling around the end points of the common edge, lines of traditional patterns match together and form a regular pentagon (Figures 20 and 21):

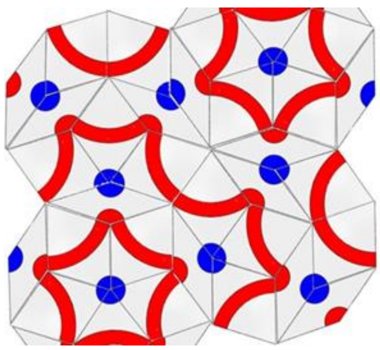

**Figure 20: Completing pattern of Figure 19**

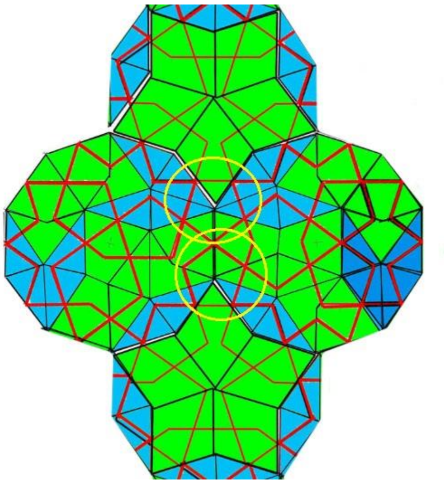

**Figure 21: Completion of traditional patterns in intersections**

By combining the two previous lemmas, we can transform any Penrose tiling with thin and fat rhombuses to a tiling by Islamic motifs (i.e., Sormedan, Panj, Tabl, and Toranj); it is enough to cover Penrose tilings with decagons of types A and B, and then draw the traditional pattern on each of decagons. To prove that the resulting tiling is not periodic, we need another lemma:

**Lemma 3)** If we rotate a periodic tiling with a multiple of $\frac{2\pi}{5}$ with respect to an arbitrary point, $O$) then we will have mismatches with the original pattern in infinitely many points.

Proof: The tiling is periodic, so by definition of periodicity (Grunbaum & Shefard, 1990), there are two non-parallel vectors $\vec{v}$ and $\vec{u}$ such if we transform the tiling by each of these vectors, the resulting tiling will match with the original one. These two vectors naturally define a coordinate system in which the axes are lines crossing $O$ and parallel with $\vec{u}$ and $\vec{v}$. In this coordinate system for finding coordinates of a point $A$, we draw lines from $A$ parallel with axes. Intersection points of these lines and the axes determine coordinates ("length" and "height") of $A$ (Figure 22). Let length of unit on the x-axis to be $|\vec{u}|$.

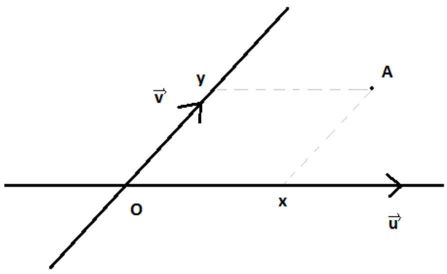


**Figure 22: Choosing the coordinate system**

The vectors $\vec{u}$ and $\vec{v}$ naturally define a parallelogram region in the tiling and the whole tiling can be considered as a tessellation of this single region. We call this parallelogram $P$. Let 0 to be the lengths of all endpoints of line segments (=verteces) of the tiling, which are in $P$. . Then length of all verteces in the tiling modula 1 are equal to $a_1$ or …or $a_n$. Let $P' = R_{O,\frac{2\pi}{5}}(P)$ (result of rotating $P$ with 72 degrees centered at $O$). Assume that there is a parallelogram region $P$ such that $P' =$

$R_{O,\frac{2\pi}{5}}(P)$ fits on the original tiling. Notice that $\overrightarrow{u'} = R_{O,\frac{2\pi}{5}}(\vec{u})$ is a transformational symmetry for the rotated tiling. Then, if $0 \leq b_1 < \cdots < b_m$ are lengths of vertexes (of the original tiling), which are in $P'$, then set of length of all vertexes of the original tiling should be $L = \{b_i + k\cos(\frac{2\pi}{5}) : k \in \mathbb{Z}, 1 \leq i \leq m\}$ (because the projection of $u'$on the x-axis has length $\cos\left(\frac{2\pi}{5}\right)$ and if we transform $P'$ by $ku'$, length of vertexes, then it adds by $k\cos\left(\frac{2\pi}{5}\right)$). But $\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}$ is an irrational number. Then $L \ (mod\ 1)$ is a dense subset in the interval $[0,1]$ and it contradicts our previous result (that this set is finite and equal to $\{a_1, \ldots a_n\}$). Then each parallelogram region $P$, $P' = R_{O,\frac{2\pi}{5}}(P)$ has at least one mismatch with the original tiling, so we have infinitely many mismatches.

**Lemma 4**) After covering a thin and fat rhombus Penrose tiling with decagons of types A and B, we can put traditional patterns (Figures 16 and 17) on them such that the resulting tiling with the traditional motifs, has a global fivefold symmetry except on one decagon which is in the center of the tiling.

Proof: Penrose tilings have a global fivefold symmetry. So, if a decagon appears in the tiling (except the central tiling), there should be four more decagons (of the same type), which are rotations of that decagon respect to the center and with an angle of 72 degrees (Figure 23).

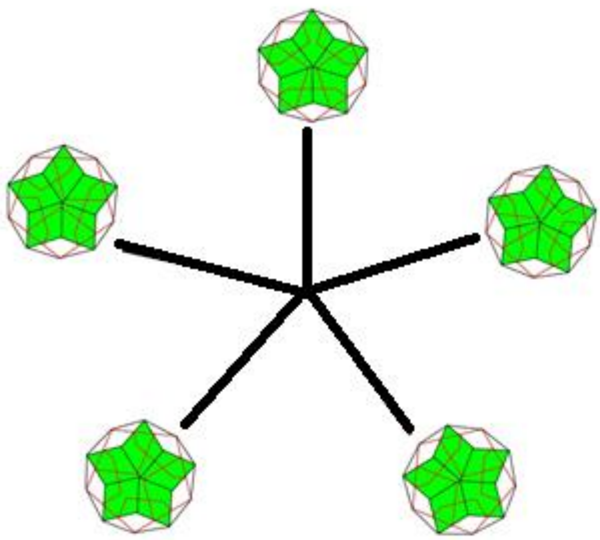

**Figure 23: Rotations of the main motif**

Case 1) If a decagon is of type A, then if we rotate a type A decagon around its center with a multiple of 72 degrees, traditional patterns, which are in five thin rhombuses in sides of decagon (blue tiles in Figure 17) do not change. Then we can choose traditional patterns in a way that a fivefold symmetry holds in Figure 23.

Case 2) If a decagon is of type B, then we can put traditional patterns in this decagon in two ways, which are the mirror reflection of each other (Figure 24):

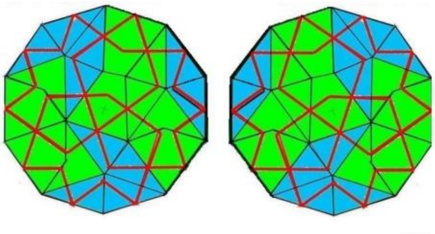

**Figure 24: Mirror reflections**

So, choosing each case does not change the traditional patterns in parts that may be in overlapping region (i.e., a thin rhombus or an Almond-six) with adjacent tiles, then by choosing one of them and then rotating that by multiples of 72 degrees, we can keep the fivefold symmetry again. ■

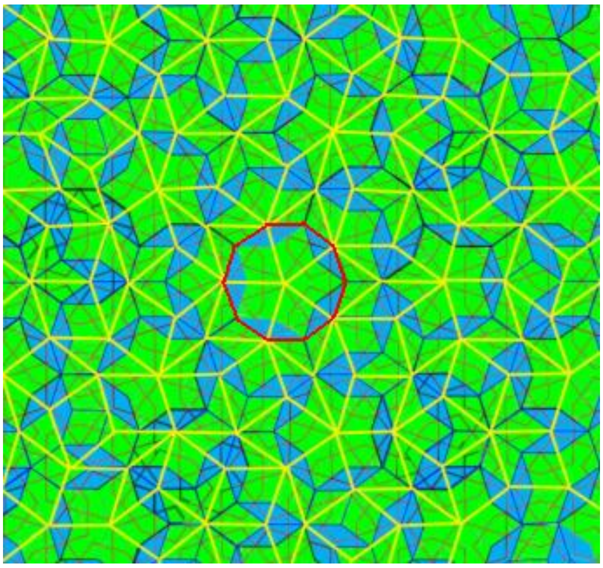

**Figure 25: Fivefold symmetry of a traditional pattern except the central decagon**

If this tiling was periodic, then by Lemma 3, we should have mismatches in infinitely many points. But this tiling has mismatches only in a finite number of points (vertices of the patterns inside the central decagon). Then we have proved the following theorem:

**Theorem1)** The tilings with traditional motifs (described above) are not periodic.

In Figure 26, one can see a section of this tiling without the background:

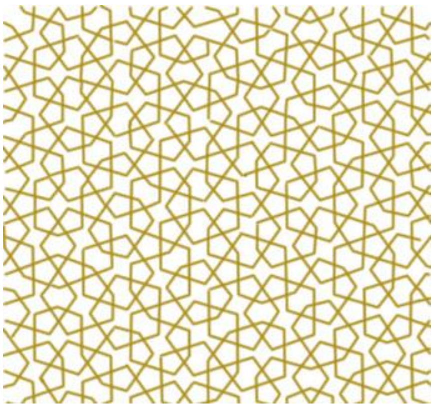

**Figure 26: Tiling resulted by our method without background**

## Summary and conclusion

As our result (and also results of Lu (2007), Rigby (2006) and others) shows, there are some relationships between Penrose tilings and Iranian-Islamic tiling based on regular decagon, which are surprising and wonderful. However, we find no evidence that non-periodicity has been noticed by the medieval artists who created these tilings. It is true that we can construct such structures by traditional motifs, but in the long tradition of Iranian-Islamic art, symmetry was always a major interest among the architectures and non-periodicity has not been interesting for them. We think that what has been interesting for designers of patterns like Darb-e-Imam, was the self-similarity aspect; a fixed set of tiles can create two similar tilings; one in a smaller scale and another in a larger scale, and maybe this fact has had a symbolic interpretation for the designers.

Another issue that is maintained in Lu and Steinhart (2007) is that a complex pattern like Darb-e-Imam has not been designed directly by compass and straightedge, but by tessellation of the, so-called, Girih tiles. We think it is true because the first author in his collaboration with master of traditional architecture, Mr. Gh. Aghaebrahimiyan in 1980s (Eslampanah, 1996), learned a method for constructing these patterns, which uses the tessellation approach, instead of direct drawing by compass and straightedge. This method is described as follows:

All motifs that are used in a kind of patterns based on regular decagons (in traditional terminology "Kond-e-Dah") can be drawn inside a regular decagon (Figure 10) and then the whole pattern can be constructed by tessellation of this motifs around the central decagon.

This fact and the existence of a large variety of different tessellations for these tiles shows that the traditional artists had discovered, in addition to self-similarity properties of these tiles, their interesting ability to match tilings together in many different ways and make different tilings, including some tilings which have no obvious periodic pattern and look like non-periodic tilling. Some examples, all from Isfahan, can be seen in Figures 30–33.

**Figure 27: Jāmeh mosque, Isfahan (photo by authors)**

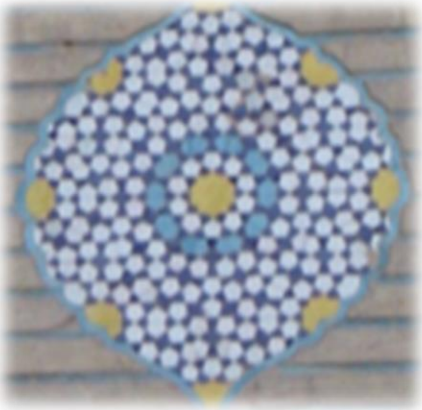

**Figure 28: Nimavard school, Isfahan (photo by authors)**

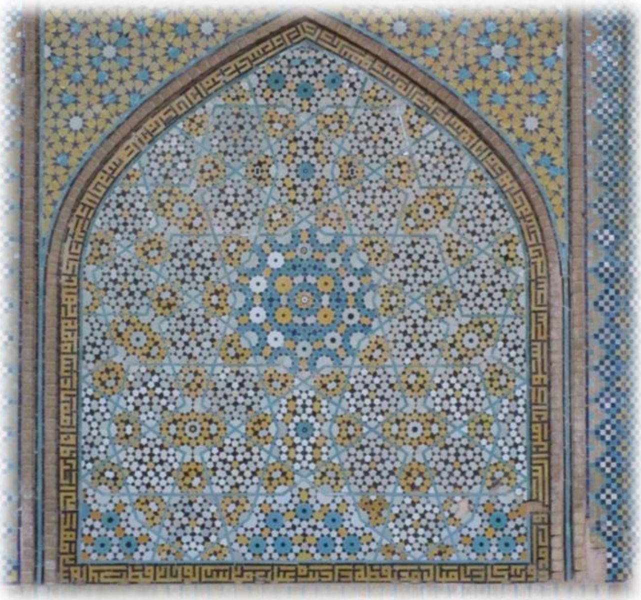

**Figure 29: Mohammad Jafar Abadei mosque, Isfahan (photo by authors)**

**Figure 30: Jāmeh mosque, Isfahan (photo by authors)**

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# John Keats's "ode on a Grecian urn" in aesthetic geometry with inversion

Revolt Pimenov[1]

*This article is about a new understanding of inversion or circle symmetry. We show pictures and visual art designed in software and write about using them in mathematics and teaching mathematics. We also present them as a basis for some methods and questions of interest for interdisciplinary education and research. In the spirit of STEM with art (STEAM), we give some examples of practical work conducted in school, as part of the Aesthetic Computer Laboratory. Mathematically, we cover some models of non-Euclidean geometries, based on circles, and relate them to emerging biological forms in these visual art constructions.*

*Keywords: Aesthetic, geometry, symmetry, laws of inversion, interdisciplinarity.*

## Introduction

> "Beauty is truth, truth beauty, – that is all [we] know on earth, and all [we] need to know." (J. Keats, Ode on a Grecian Urn)

So, the poet said, and here we will interpret his words geometrically with algorithms, illustrations, and theorems.

The material we offer combines aesthetics, geometry, and topics of higher mathematics. We believe that this material can be used (and has actually been used) in work with children and students in schools. What is aesthetic geometry and how can we use it for mathematics and teaching mathematics? It is a unified approach to geometry based on the symmetries of circles. We all know that a circle is the most symmetrical figure, but there is also an aspect of symmetry about the circle, which we can define using simple and non-traditional methods that can lead to a variety of opportunities.

We will start by introducing the aesthetic part, paintings and animations or visual art, and then explain the basics of geometry and pedagogical applications.

## Paintings

All pictures in this section were made in a graphic editor, *CorelDraw*. We used special macros that implement the geometry of the circles. We have countless images made in this way but show the first three to explain the variety of emerging art forms, followed by the fourth, which clarifies their geometrical essence.

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**Figure 1: “Orchestra in the air”**

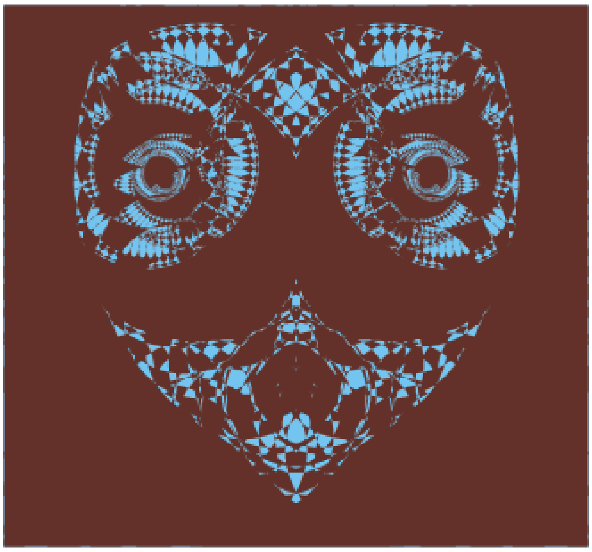


**Figure 2: “The face of the grotesque”**


**Figure 3: “The bird of Galactic and Peace”**

The unusual, two-centered baroque spiral (Figure 4) was obtained by the composition of inversions. From the parts of such spirals, the first three drawings were created. From a mathematical point of view, this spiral is an excellent introduction to group theory and illustrates the concept of a limit.

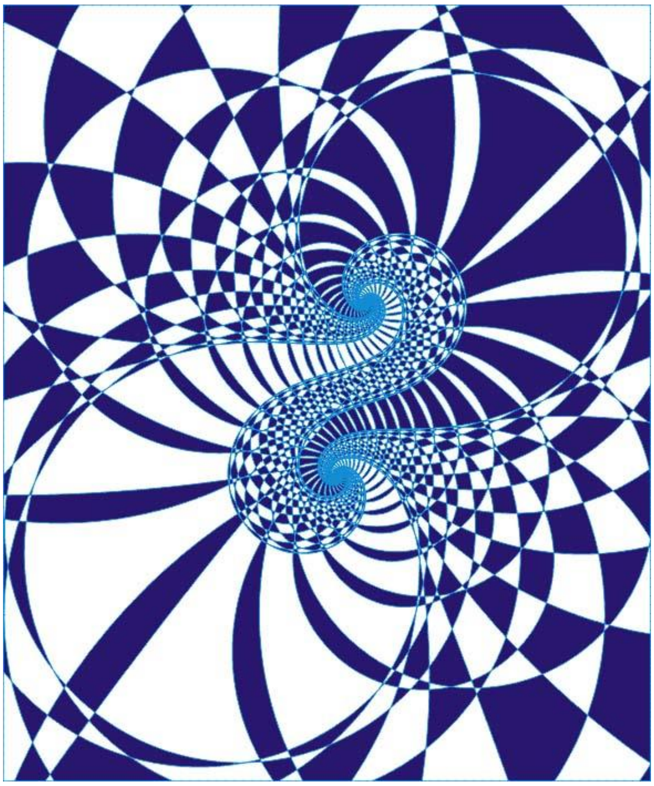


**Figure 4: "Baroque spiral"**

## Visual art

Visual art presented in the previous section consists of endless combinations of graphical components whose construction follows the language of mathematics. The laws of inversion create an environment in which each change is harmonious. In this environment, we insert objects, and they move without repeating and without ending, remaining harmonious. Time and rhythm are part of this harmony. We can modify pictures by changing colors and moving objects, producing an effect similar to the movement in a force field. It is an example of the intersection of aesthetic geometry with physics, which we can use to model the movement of planets in the Solar System. While the graphics are dynamic, they constantly change and evolve, the examples we give come from frozen frames (you can download a program for viewing visual art by following the references in the last section). The composition of circles' symmetries is the foundation of all pictures.

I used a special program, *DodecaLook* (Dodeca Meditation, n.d.), for a demonstration of this art. The questions that I ask myself and pose to the readers are: Why are the forms that are created by the laws of circles' geometry beautiful? And why are these forms often zoomorphic (e.g., Figure 5)? I did not intend to create them in such a way, such results stem from the inner properties of circles' geometry, our aesthetic sense, and biology. These create an opportunity not only for teaching but also for research.

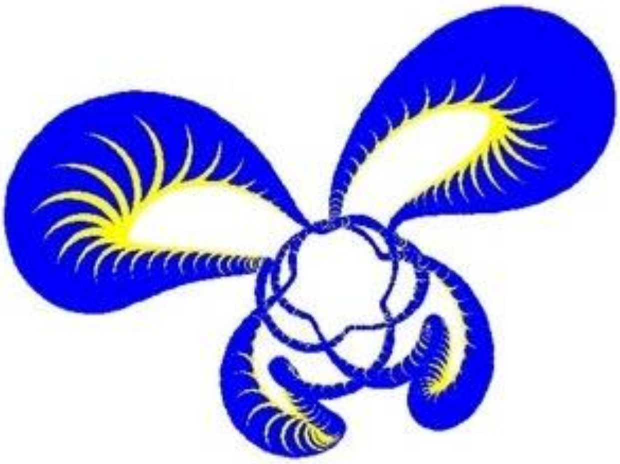


**Figure 5: “Flying knot”**

In some cases, the program does not create endless pictures in the time combination of images, but knots (Figure 6), and there is an interesting theorem about this phenomenon.

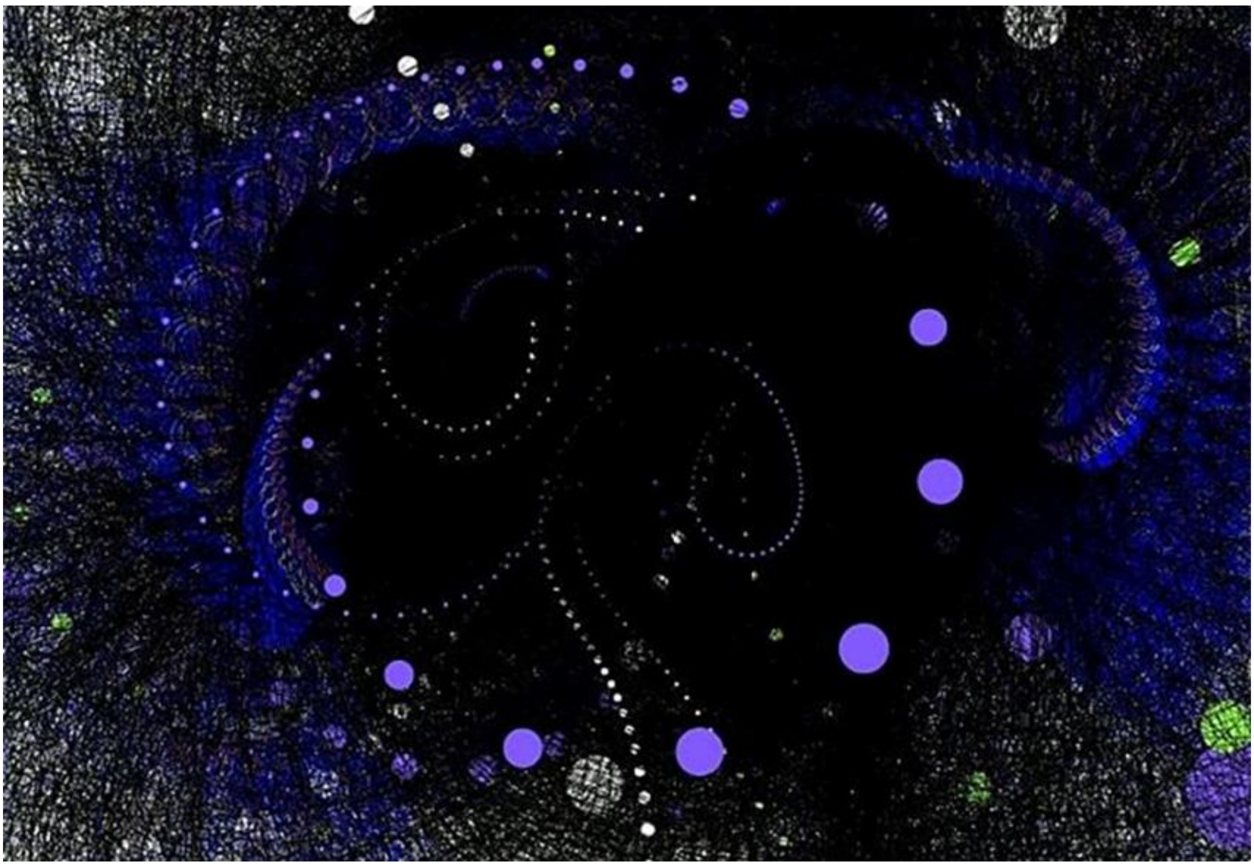


**Figure 6: “Cosmos”**

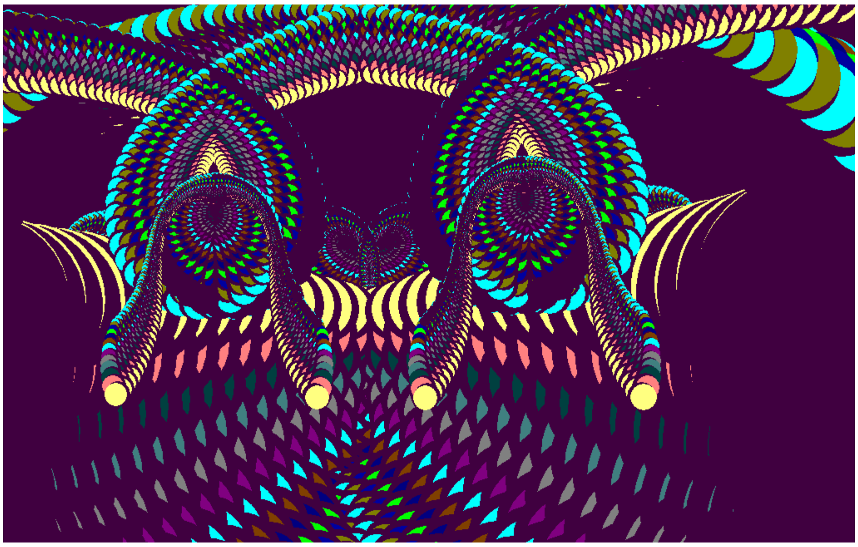
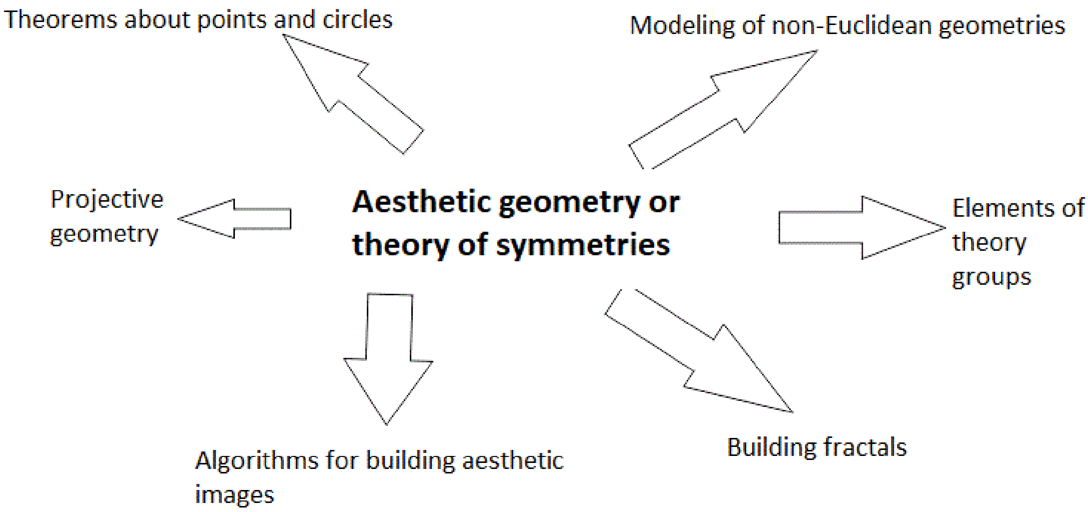


**Figure 7: "Mourning"**

The software *DodecaLook* is good for developing an aesthetic sense in mathematics students, and for playing with colours and forms.

## Mathematics applications

In this section, we show topics of mathematics that are convenient to study with inversion, or, as we say, with "Aesthetic Geometry."




*http://bogemnyipeterburg.net/revolt/matem/teachpictures/index.html*

**Figure 8: Aesthetic geometry connections to other mathematical topics**

The ideas expressed in this section, are close to Friedrich Bachmann's (1973) book, "Building geometry from the mirror concept." Symmetry considered in this book is symmetry about a straight line, while we consider symmetry about circles, which creates amazing new possibilities. The next sub-section provides the main idea about symmetry relative to circles and about modelling non-Euclidean geometries.

## Definition of inversion with two pairs of symmetrical points. Pairs of the pairs

I recall the classic definitions of a circle. The first definition uses a formula; it states: "If we have a circle S with center O and radius r, and point X, then a point obtained after inversion about S (let's denote it S(X)) is the point Y that lies on the beam OX, such that $|OX|\cdot|OY|=r^2$."

The second definition is more geometrical, it states: "Let X is outside of S, then create two tangents on S from X, and create a chord through the points of their contact with the circle. Now pass the line through O and X and name the point of intersection of the chord and this line, Y=S(X). If X is inside S, we do this construction in reverse order; construct a line OX and a chord through X that is orthogonal to this line, and a pair of tangents on S in the points of intersection of the chord and S. The point of intersection of these tangents will be image of X, S(X). This defines a symmetry of a point about circle S which is involution: S(S(X))=X. The inversion transforms circles to circles or lines, and we say here that a line is the special case of a circle; the circle with an infinite radius! Now we can say that inversion is symmetry in the world of circles.

These definitions of inversion have defect because in these definitions we use straight lines, centers of circles or distance between points, but the words "centers," "straight lines," and "distance" are not from the world of circles. They are not invariant under inversion. In the world of circles, we can only draw a circle through 3 points, make inversion and find points of intersection between circles. However, another definition and understanding of symmetries between circles exists, it is marvelous that such a definition is easy, easier than the classic definition, and easier than the definition of symmetries about line. The remarkable thing about this definition of inversion is that it only uses elements that are invariant under inversion!

Let us have two pairs of symmetrical points: A and B are symmetrical under symmetry I, I(A)=B, and C and D are symmetrical about I, I(C)=D. In that case, as is well known, all 4 points must lie on one circle; it is the circle S (Figure 9). Let us then have some arbitrary point X and find a point symmetrical to it, point I(X).

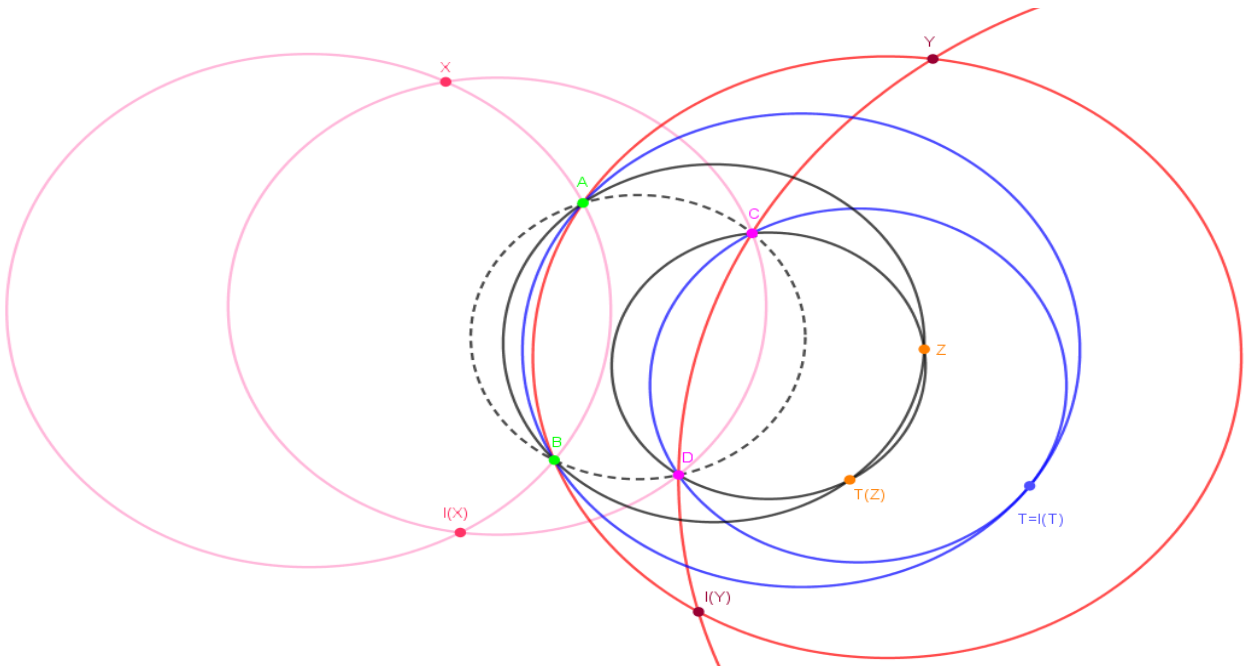



**Figure 9: "New Definition of Symmetry"**

For this we must do only two operations, we must draw one circle through X and the first pair of symmetrical points A and B, and another circle, through X, C, and D. The second point of intersection of these two new circles is a desired point, symmetrical to X under symmetry I (i.e., I(X)). Following this reasoning, perform analogous operations with points Y and Z. If the two new circles will tangent each other (have only one common point), as it is in point T, we say that T=I(T) and T is unmoved, fixed point under I. The set of all fixed points under I, is a fixed circle. We usually define the symmetry I, starting from this circle.

After this, we can formulate nice theorems about points A, B, C, D after dividing them in pairs (as pointed out in ample literature), this definition and theorems could be a good introduction to group theory.

## Threecircler and a royal way to non-Euclidean geometries

We call three intersecting circles, a “threecircler,” by analogy with a triangle. The pairs of points of these circles’ intersections define a symmetry; under it, each of the three pairs of points change place with each other. This definition is more geometrical than the definitions we presented in the previous section, but the result is the same. We named this symmetry, a “threecircler’s symmetry,” since it is very important for the circle geometry. There is no analogy in the geometry of a triangle for such a symmetry. But we have a good analogy for many other important properties of triangles and threecirclers where we think of a pair of points intersections as a triangle vertex and circles (or arc) as sides of a triangle. In fact, we have here an absolute geometry.

Let us classify threecirclers by a property of division separation. We have three possibilities: one circle divides the pair of points of intersection of the two other circles, or the pair of points of intersection lies on one side of the third circle, or as an intermediate case, one point of intersection of the two circles lies on the third circle. Let us call threecirclers of the first type, “Riemann threecircler” (or “elliptic threecircler”); the second type, “Lobachevsky threecircler” (or “hyperbolic threecircler”); and the third type, “Euclidian threecircler.” We see the Riemann threecircler and the Lobachevsky threecircler in Figures 10 and 11.

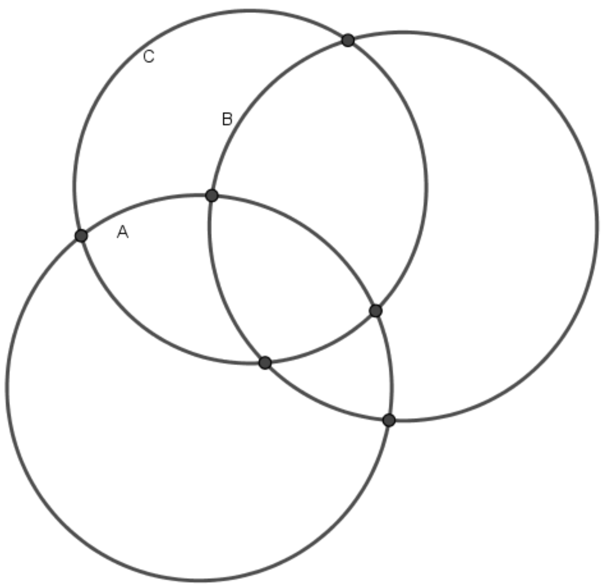



**Figure 10: Riemann threecircler**

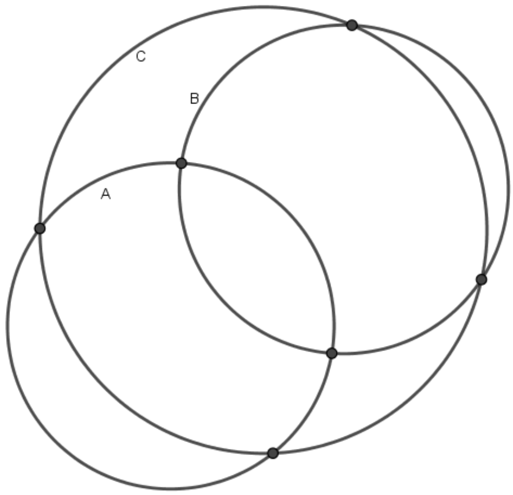



**Figure 11: Lobachevsky threecircler**

Let us consider the pairs of points of intersection as vertices and circles as sides of a threecircler. We can then define bisectors, altitudes, and medians in a threecircler in the same way as in a triangle using inversion and orthogonal circles as we use the symmetries about lines and orthogonal lines in Euclidean geometry. Furthermore, we can use angles between circles as we use angles between lines in Euclidean geometry.

In Figure 12 we see the intermediate case when all three circles pass through one point. It is the Euclidean case, since after inversion with the center in this point, the threecircler transforms to a usual triangle and all properties of this triangle are given by the threecircler. From a threecircler, as an 'absolute triangle,' we can easily develop an absolute geometry. Let us have some threecirclers and let us draw any circles through pairs of points of its intersections, we can call these circles as the "lines of absolute plane." The pairs of points of intersections we call "points of absolute plane." If two drawn circles intersect, we call the pair of points of intersection "absolute point," as we did above.

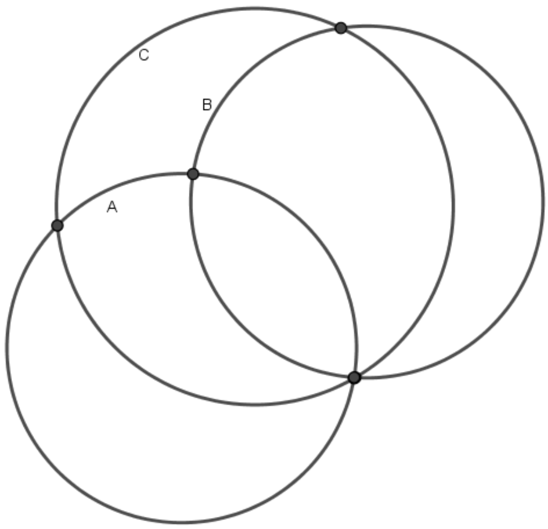



**Figure 12: Euclidian threecircler**

The type of a threecircler will define the type of created plane: Riemann, Lobachevsky, or Euclidean. All needed theorems can be easily proven. We can model projective geometry also by using the circle geometry, as it is documented in the literature.

## Pedagogical implications: Aesthetic computer lab

There are many possibilities to use this approach in pedagogical applications for all ages. For example, we could engage with forms generated by such methods, we could teach some main mathematical ideas using such illustrations, or dance following the animations of the visual art. This topic is a good introduction to the theory of groups (e.g., for permutations, continuous, and Lie groups).

The example in Figure 13 illustrates three mathematical concepts: limit, action group on a set, and composition of transformations. After we replace each point with a circle, we create a beautiful necklace, which turns a mathematics lesson into an aesthetics lesson. The first example is in Figure 13, but any baroque spiral (Figure 4) is a more complex and a more beautiful example.

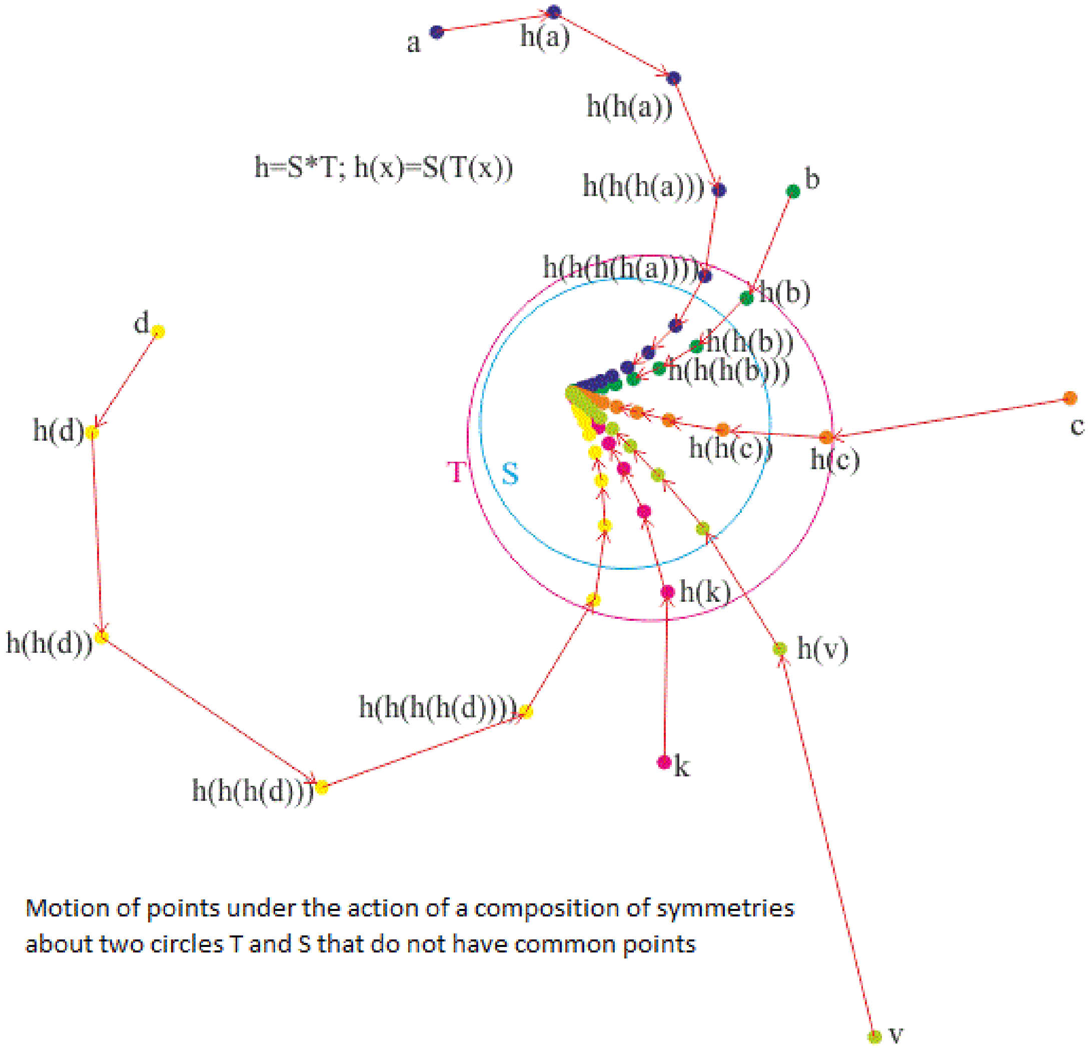



**Figure 13: Motion of points**

Below are two examples of practical work with students.

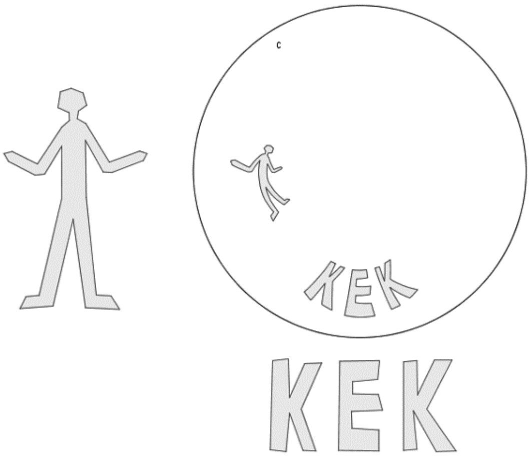



**Figure 14: Student creation N#1**

Elements of aesthetic geometry were taught in the lessons and electives at the Alferov Physics and Technology Lyceum, in St-Petersburg. Figure 14 is an example of students' work; when schoolchildren first learned about inversion, they drew a man and named him Kek. After an inversion, Kek danced, which was engaging for students and a clever use of play in mathematics classroom. Figure 15 is another example of student work made after some lessons in a computer lab; the work is in the spirit of STEM because creating such baroque spirals is an intersection between computer programming, geometry, and engineering. It is STEM with aesthetics! Concepts such as "limit", "algorithm," and "action transformation on set," are all involved in this work.

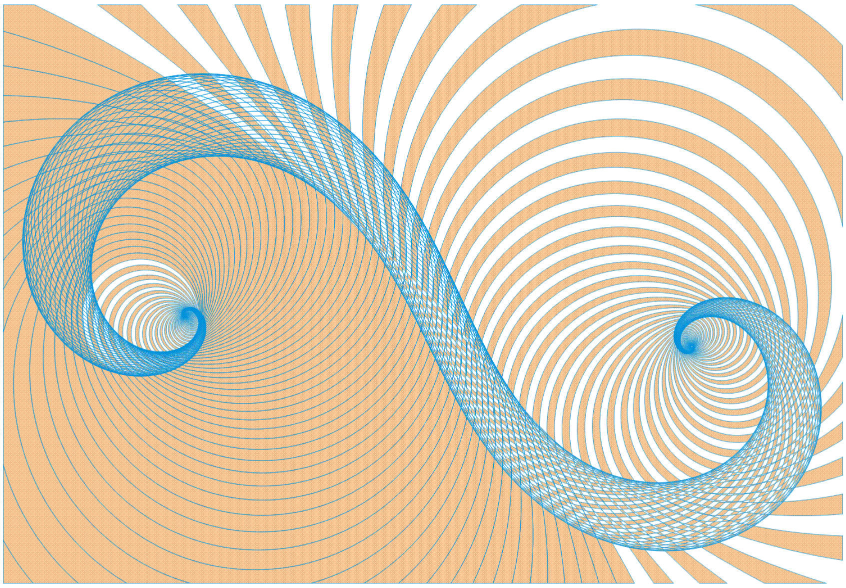


**Figure 15: STEM with aesthetics**

We have other examples of student work, one animation created in GeoGebra illustrates Steiner's porism about circles tangent to two given circles. This was created without using straight lines or centers of circles. It helped the students develop a good understanding of the world of circles where lines and centers do not exist.

We are also aware of a 4-year work in the school №96 "Eureka Development," in Rostov-on-Don, about aesthetic geometry.

## Interdisciplinarity

In the previous sections, we have shown how aesthetic geometry can turn mathematics lessons into design or drawing lessons (Pimenov, n.d.). Now we will turn to the amazing and fascinating connection between aesthetic geometry and biology. When we create spirals like in Figure 5, we often see zoomorphic forms (e.g., Figure 4). If we start playing with the pieces of these spirals, both anthropomorphic forms and forms reminiscent of ancient art appear. Anthropomorphic forms also arise during the work of animation art. We cannot predict when and how such forms arise, but they do. This, we believe, suggests that biologists may also benefit from an introduction to aesthetic geometry. Perhaps this approach may shed new light on morphogenesis? What we refer to here applies not only to pedagogical activity, but also to research work.

As an example, we show a drawing of Cheburashka, a character from a popular cartoon, that emerged from spirals. We think that the biological component of aesthetic geometry is noticeable in other drawings as well. We suggest that in the world of circle geometry, there exists an enigmatic connection between the inner, non-visual beauty of abstract mathematics, and the outer beauty of the world that surrounds us. This realization is very fruitful and can lead us to marvelous new results, as well, it can help to fortify the integrity of education and importance of teaching some mathematics concepts that are not strictly curriculum based.

**Figure 16: Cheburashka**

## Acknowledgements

I thank Dragana Martinovic for her help in editing this article and for her attention to aesthetic geometry. I also thank all MACAS.

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# Dancing mathematical processes: Stochastic and flow of motion dance

Jorge Soto-Andrade[1], Ami Shulman[2] and May Garcés-Ocares[1]

*We explore, theoretically and experimentally, the interplay of mathematics and dance, focusing on dancing mathematical processes (instead of mathematical structures). We engage then in stochastic dance and flow of motion dance. Stochastic dance arises from the fact that randomness generates shapes and forms, and also movements, dance and even choreographies. We approach stochastic dance as an enactive and experiential analogue of stochastic music, where the poetry of a choreographic spatial/floor pattern is elicited by a random walk. We exemplify flow of motion dance as the choreographic enaction of arithmetical dynamical systems in the finite cyclic universe of the integers modulo m. We report and discuss the lived experience of a group of professional and amateur dancers in a recent pilot workshop, for whom dancing mathematical processes triggered a reconstruction of their relation to mathematics.*

*Keywords: Stochastic dance, flow of motion dance, random walks, arithmetical dynamical systems, metaphor.*

## Introduction

We intend to explore, from a theoretical as well as an experimental way, one facet of the relationship between mathematics and art. More precisely, we focus on the interplay of mathematics and dance. For related work on mathematics, movement and dance, see Gerofsky (2013, 2016, 2017), Gerofsky and Ricketts (2014), Milner et al. (2019), and Pastena et al. (2013).

We need, however, to first address the question: *What is mathematics about?*

The dominant ideology in mathematics (Bourbaki, 2006) says *Structures!*

So, mathematics is the science of structures. But "Structure" is just a metaphor. As usual, as the old Persian proverb reminds us: "The fish is the last one to see the water," or the metaphor.

We claim in a dissident vein, that mathematics is rather about *processes*!

Nowadays, indeed, compositionality, diagrammatic thinking, string diagrams, deterministic processes (dynamical systems), and stochastic processes (random walks) are ubiquitous in mathematics.

Interestingly, stochastic processes can be metaphorized as deterministic: when we watch a random walker, we may "hallucinate" seeing it *splitting* instead of choosing randomly its direction. For example, we may metaphorize a frog jumping symmetrically (like tossing a coin to decide between right and left) on a row of stones in a pond, as a frog, which splits evenly into two halves, instead of

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jumping each time. The random walk becomes then a splitting process, enabling us to *construct* the concept of probability by this metaphorical sleight of hand! Indeed, the probability of finding the frog at a given stone, after *n* jumps is just the “portion” of the frog you find there after *n* splittings.

It could be argued that this duality: *structure* versus *process*, is intrinsic to mathematics and analogous to the *particle-wave* duality in Quantum Mechanics.

So, from an enactivist viewpoint (Varela et al., 1991), the question: “Which is the true nature of mathematics?” is as naïve, or non-sensical, as the analogous question about the true nature of the electron.

In our view, the same duality arises in performance and choreography in dance. In choreography, we have “structural aesthetics,” usually aiming at helping the watcher to “see time in space.” The performance, or interpretation, of the dancer, appears as a process.

The well-known Beckett’s Quadrat[3] may be seen as a beautiful blending of structure and process. Notice how the bent monk-like posture of the walkers emphasizes the process of walking from here to there and so on, like a forward pointing arrow. A more upright posture would generate a completely different choreography.

Structure may be seen as a constraint, but it is also a background, or a playground, where freedom may be enacted.

In our approach, the dancers enact processes instead of building structures with their bodies. So, they dance *mathematical processes* (instead of dancing *mathematical proofs*, for instance, as in the related work of Milner et al., 2019). We engage then in *stochastic dance* and *flow of motion dance*, which we describe below.

## Stochastic dance

*Stochastic dance* is akin to stochastic music, developed last century by Xenakis. It has however, older avatars, like Mozart, who showed how to compose minuets by tossing dice. Similarly, contemporary choreographer Cunningham took apart the structural elements of what was considered to be a cohesive choreographic work (including movement, sound, light, set, and costume) and reconstructed them in random ways.

It is well-known that randomness generates shapes and forms: tossing a coin one hundred times generates a fractal-like shape resembling a mountain ridge. Our main idea is that randomness also generates movements, dance, and even choreographies.

To try to embody this idea amounts to exploring an enactive and experiential analogue of stochastic music in the realm of dance, where the poetry of a choreographic spatial/floor pattern is elicited by randomness in the form of a mathematical stochastic process, e.g., a random walk—a stochastic dance of sorts.

Among many possible random walks, we consider two simple but paradigmatic examples (Soto-Andrade & Shulman, 2021), in the form of two scenarios proposed to the students/dancers:

*First scenario*

[3] Text und Bühne. (2012, November 12). Samuel Beckett: Quad I+II (play for TV) [Video file]. https://www.youtube.com/watch?v=4ZDRfnICq9M

The walker is a frog jumping randomly on a row of stones in a pond, starting at a given stone and choosing right or left as if tossing a coin.

*Second scenario*

The walker is a person walking randomly on a square grid, starting at a given node, and choosing each time randomly, equally likely N, S, E or W, and walking non-stop along the corresponding edge, up to the next node, and so on.

Quite natural, but somehow "impossible," questions arise in this situation, like:

*Where will the walker/dancer be after a while?*

Notice that we have no definitive answer to this question. The first answer that comes to mind is: "Nobody knows." We may, however, notice stepwise that after a given number of jumps or steps, there are possible and impossible locations, some possible locations are "more possible" than others, etc.

Mathematically, however, these random walks may be studied in a friendly and efficient way with the help of "hallucinatory" metaphors (Seth, 2021; Soto-Andrade, 2014, 2018), e.g.,

*A Splitting Metaphor: The walker splits equally into parts instead of randomly choosing its direction of movement.*

*A Pedestrian Metaphor: Instead of our single walker, we hallucinate a bunch of walkers, which split equally at each stage into subgroups which go in the different possible directions.*

These metaphors suggest several ideas for a choreography, which are more complex than just having one or more dancers perform the random walk, and which turn our random process surprisingly into a deterministic one!

We now discuss in more detail our two scenarios, both mathematically and choreographically, with the help of the aforementioned metaphors.

## First scenario: The random walk of the frog on a row of stones in a pond

The *Splitting Metaphor* sees that the frog, instead of jumping right or left, splits into two halves, which go right and left.

The *Pedestrian Metaphor*, in the case of a 3-step random walk, sees an army of eight frogs crouching at the starting stone, which splits into two groups of four, going right and left, which in turn, split into two groups of two, going right and left, which in turn split into two single frogs, going right and left. See Fig. 1 for a mathematical visualisation of this process *unfolding in time*, where time flows downwards, as indicated by the vertical descending arrow. Here the horizontal green path of the frog corresponds to the choices: (our) right, left, left. The descending green path is the unfolding in time of the actual frog's path (as in relativity theory).

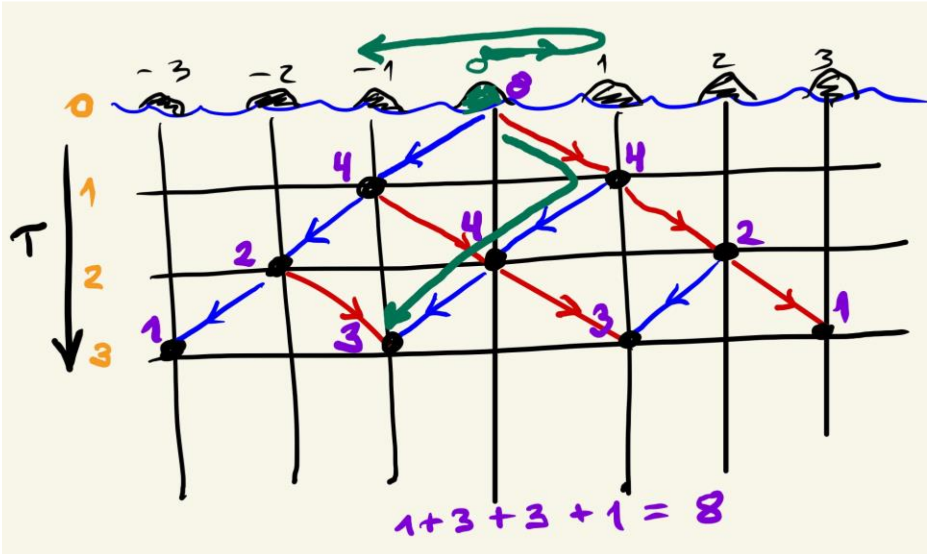


**Figure 1: The 3-step random walk of the frog**

As we see in Figure 1, the successive distributions of the army of eight frogs may be described as

- (0,0,0,8,0,0,0)
- (0,0,4,0,4,0,0)
- (0,2,0,4,0,2,0)
- (1,0,3,0,3,0,1)

We enacted this simple choreography in a workshop with eight dancers on October 17th, 2022. In this video[4], we can see the eight frogs/dancers splitting thrice into halves on the line of seven stones/spots and so executing each of them *one of the different eight possible paths of the frog*.

In fact, in our workshop, we just asked the eight dancers to enact, each of them, one of the eight possible paths of the frog. In other previous workshops, with several groups of eight participants working in parallel (mostly in-service primary school teachers), we observed that some groups chose a “Magister Ludi,” who would hand out a script to every dancer, indicating the path (e.g., right, left, left), which the dancer is to enact. Other groups, however, after some discussion, proceeded in a clever, non-hierarchical way, dispensing with a Magister Ludi by just splitting thrice into halves! In our last workshop, we observed that at least one dancer spontaneously got the same idea.

After arriving at their end node/position, the dancers can try to retrace their steps to return all to the starting node. This creates a quite harmonious unfolding and enfolding pattern, which could be repeated over time.

Of course, for a 4-jump random walk, we would need to recruit sixteen dancers starting at the same node of a discrete line on the stage and execute, each one, a different path of the sixteen possible 4-jump paths the frog can follow.

Notice that among other possibilities, the dancers could also enter the stage, one by one, describing sequentially the different possible random paths in the spirit of Beckett’s Quadrat[1]. However, to have

[4] Maycenii, G. (2022, November 15). Dance/math workshop (3) [Video file]. https://youtu.be/7m7d3kNxPx4

a closer analogue to Beckett's choreography, we could let our frog jump on a *square* (or another polygon) of stones in a pond. Then we could have 16 dancers execute each of the sixteen different possible 4-step paths the frog can follow, starting at the same stone, and exiting the square after the fourth step, with a time shift of two steps, say. So here, the dancers instead of following all the *same* deterministic path, with a time shift, as in the Quadrat, would be following a different deterministic path each.

## Second scenario: The 2D symmetric random walk on a grid

The *splitting metaphor* hallucinates the walker splitting into four-fourths, which go N, S, E, W at each step, and so on. Then the portion of the walker landing at each case after *n* splittings (obtained by merging the incoming minor pieces of the walker) would give the *probability* of finding the whole walker at that location after *n* steps.

In our workshop, we enacted, instead, the friendlier *pedestrian metaphor* for a 2-jump 2-dimensional random walk, which hallucinates a group of 16 walkers, which split evenly into four groups of four, which go N, S, E, W, then split again into four walkers, each going to each cardinal direction. See Figure 2 for a visualization of this metaphor, where the 16 walkers-dancers are represented by stacked coloured cubes, like Chinese dancers, who could climb onto each other's shoulders. Notice that the groups of four have split sequentially, one after the other, instead of splitting simultaneously, as in the actual choreography, for the sake of clarity.

Notice that in this way, the sixteen dancers enact the sixteen possible 2-edge paths of a single random walker, so their *collective movement* would reveal then the unfolding of all possibilities for this (2-step) random walk.

Also, notice that the final distribution of the group of sixteen dancers provides a sensible answer to the "impossible question": *Where will the walker be after two steps*?

We see indeed the possible locations (nine possible cases) for the walker and that some are "more possible" than others (they have more cubes stacked on them). This qualitative ranking can then be quantified by the proportion of walkers standing a given case after walking two edges. For instance, we have 4/16 for the starting case (the home case) and 1/16 for the four "extreme" cases.

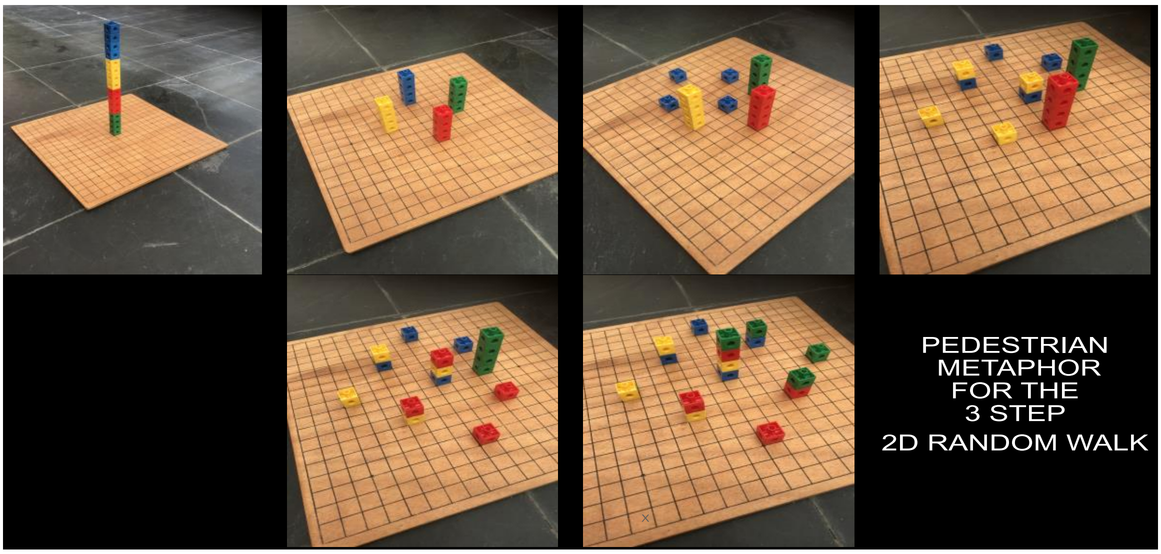


**Figure 2: Pedestrian metaphor for the 2-step 2D random walk**

In this way, we have constructed, *in a pedestrian way*, the notion of *probability*: we say that the *probability* of finding the walker back at home after two steps is 4/16, and so on, for the other possible cases. This is the best answer that we can give to our "impossible" question.

For a 3-edge 2D random walk, this metaphor would recruit a group of 64 dancers, which split evenly thrice into fourths, enacting in this way the 64 possible 3-edge paths of the single random walker.

There are many other possible choreographies suggested by this type of random walk. Among others, the dancers could choose their direction ad libitum after some spinning, each time, on a grid-free stage, but keeping the same step length, as in statistician Pearson's model for a mosquito random flight (Pearson, 1905)[5].

We are interested in various possible spin-offs of these choreographies, which intertwine dance and mathematical cognition. For instance, when the dancers choose each one a different path, they will notice that their final distribution on the nodes is uneven (interesting shapes emerge, as in Figure 2). In this way, just by *moving together*, choreographer and dancers can find a quantitative answer to the impossible question: Where will the walker/dancer be after a while? Indeed, the percentage, or proportion, of dancers ending up at each node after a given number of steps *is* the probability of the random walker landing there after that number of steps. This could be seen as an instance of *swarm intelligence*, where knowledge emerges from a collective performance. We get in this way a glimpse of the interplay between individual agency and the intelligence of the whole.

## Flow of motion dance: Dancing arithmetical flows

By *flow of motion dance,* we understand in this paper the choreographic bodily enactment of a dynamical system in some mathematical setting. As the stage for this dynamical system, we focus on a finite cyclic universe, mathematically modelled by the integers modulo $m$ (for a fixed natural integer $m$), which we visualize as a regular polygon with $m$ sides, on which the discrete line of the relative integers is suitable wound or coiled around.

We remark that usually, students do arithmetic modulo $m$ by calculating with the integers and reducing modulo $m$ only in the final step. Our idea is on the contrary, to really *move* to the finite cyclic universe of the integers modulo $m$ and to dwell and calculate there.

Our dynamical system will be *arithmetical*: its dynamics is given by the repeated multiplication by a fixed integer $k$ modulo $m$. Then each integer $a$ modulo $m$ is the starting point of a trajectory (its fate, or destiny, metaphorically speaking) consisting of its successive images

$a, ka, k^2a, k^3a, \dots , k^na, \dots$

under multiplication by $k$ modulo $m$. This trajectory is usually called the *progressive orbit* of $a$ under multiplication by $k$.

See Jorge Escuti video[6] for an animation of the orbits of this dynamical system in the case $m = 16$ and $m = 17$ for all possible $k$'s, from which Figs. 3 to 6 are extracted.

[5] Cranna, V. (2015, March 27). Karl Pearson and Sir Ronald Ross. Library, Archive & Open Research Services blog. https://blogs.lshtm.ac.uk/library/2015/03/27/karl-pearson-and-sir-ronald-ross/

[6] Escuti, J. (2020, August 10). Órbitas de multiplicación módulo 16 y 17 | Álgebra modular [Video file]. https://www.youtube.com/watch?v=KRFTIEAs75M

Notice that since our universe is finite, the trajectories of each integer modulo *m* must necessarily end up into an endless cycle (including the case of a cycle of length 1, i.e., a fixed point). A very interesting phenomenology of trajectories arises, as exemplified in the aforementioned animation.

In our choreography, we recruit *m* dancers, each playing the role of an integer modulo *m*, embodied in a vertex of our regular *m*-gon. As in a flow in a river, where each drop of water follows a definite trajectory, every dancer starting at any given vertex of the polygon will follow a definite trajectory, enacting his/her "destiny", which goes forever in a cycle or ends up falling into a sink.

This creates the visual effect of a (human) flow, where dancer 0 is always a fixed point. An interesting phenomenology emerges, especially for non-prime *m* (like 12 or 16), where we find "attractors," which can be "sinks" or "cycles" of various lengths.

For instance, for $m = 16$ and $k = 12$, we see a flow that inexorably dies out—all the dancers finally "fall" into the sink 0 (Fig. 3). The same happens for k = 6, but in a more intricate way (Fig. 4). For the case k = 13, we have two "square cycles," vaguely reminiscent of Beckett's Quadrat, besides two "flips" (2-cycles) and four fixed points. Interestingly, from an arithmetic viewpoint, we get the same flow but backward (the "reverse flow") if we take $k = 5$, for which the calculation of the flow is friendlier. The arithmetical explanation of this fact is that 13 x 5 = 65 = 16 x 4 + 1, which equals 1 modulo 16 so that 5 is the multiplicative inverse of 13 modulo 16. So, multiplying by 5 "reverses" the effect of multiplying by 13.

In the prime case $m = 17$, for $k = 4$ the global flow (fixed point 0 excluded) *decomposes* into four disjoint flows, which are "isomorphic" 4-cycles. A pattern which could be revealed through the dancers' movements (who could be clad in the corresponding cycle colour).

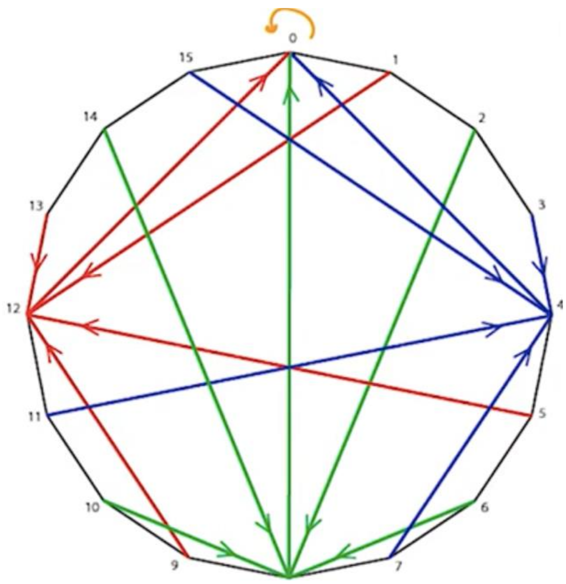


**Figure 3: The flow for *m* = 16, *k* = 12**

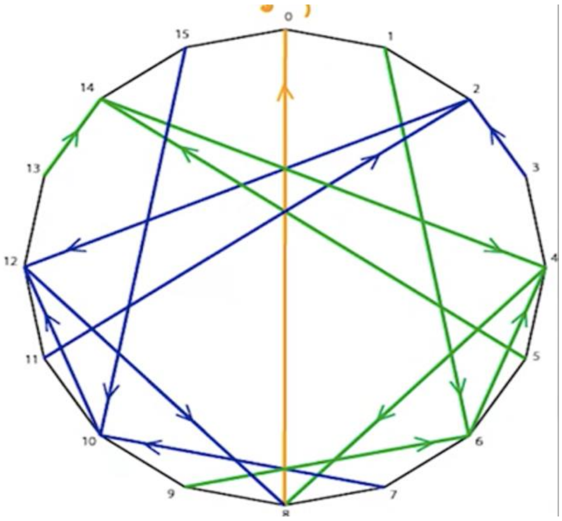


**Figure 4: The flow for *m* = 16, *k* = 6**

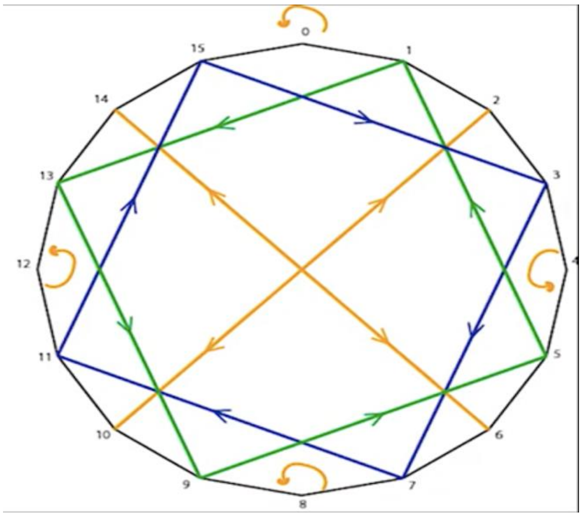


**Figure 5: The flow for *m* = 16, *k* = 13**

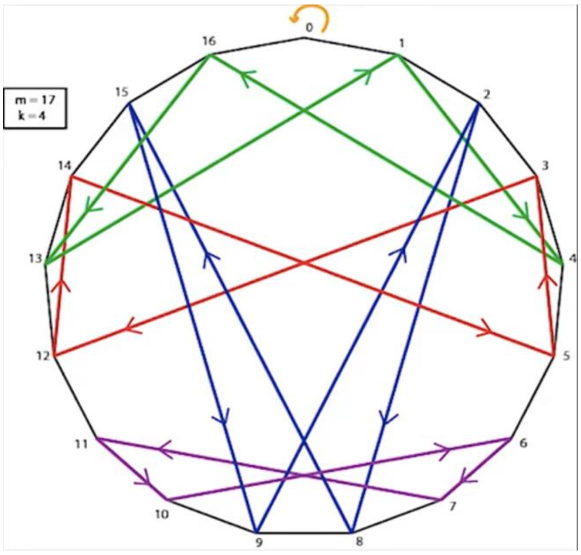


**Figure 6: The flow for *m* = 17, *k* = 4**

In our 2022 workshop, we enacted the "clock case" ($m$ =12), for $k$ = 3 only, whose dynamics is pictured in Fig. 7, drawn by one of our mathematics teacher students.

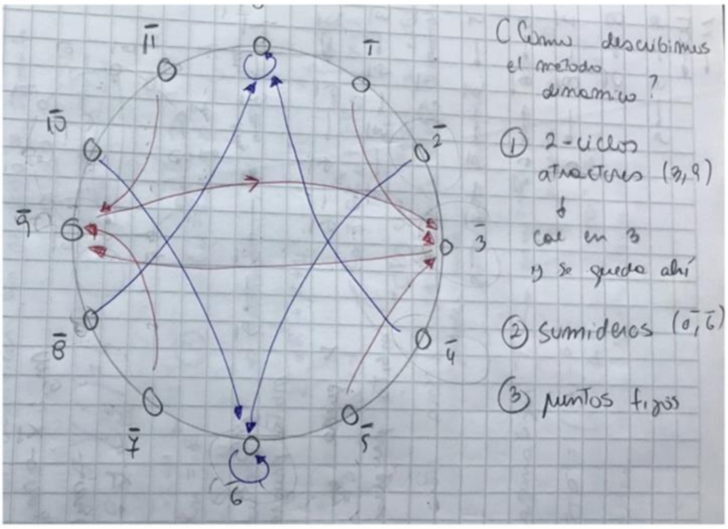


**Figure 7: The flow for the "clock case" *m=12* and *k=3***

We see in this case:

- Two fixed points, 0 and 6, which are "sinks" for 4, 8 and 2, 10, respectively.
- A 2-cycle formed by 3 and 9, which "attracts" 1, 5, 7 and 11.

This arithmetical dynamic was enacted choreographically by 12 dancers in our 2022 workshop[7]. They calculated their trajectories mentally first (e.g., 1, 3, 9, 27=3, 9, 3, …, modulo 12). Then they improvised freely the way they moved from one vertex to another, paying attention to each other's movement, instead of moving with a rigid gait as in Beckett's Quadrat.

## Findings

In our previous workshops with primary school teachers engaged in a professional development program at the University of Chile in Santiago and elsewhere in Chile, we observed that roughly one half of the groups appointed a "Magister Ludi" to enact all the different possible paths of the walker, with no repetition, and the other half realised they could proceed in a non-hierarchical way, just by splitting into halves, over and over.

In our 2022 workshop, one dancer—with no special mathematical training or inclination—immediately suggested the non-hierarchical way, in the case of the frog jumping thrice, which was readily accepted by the other dancers.

All dancers participating in the later workshop reported afterwards that their enacting of mathematical processes had been a dramatically unexpected experience. Indeed, mathematics was hitherto something totally alien to them, just formulae, calculation by rote or esoteric Greek plane geometry.

[7] Maycenii, G. (2022, November 15). Dance/math workshop (1) [Video file]. https://youtu.be/bQPGmcgCelU

Now, they felt that they could access mathematics through their bodily experience, a cognitive modality which was ignored, thwarted, and even despised in their previous educational experience.

Some said that the embodied approach to mathematics they had experienced was a powerful tool to *democratize* mathematics, unknowingly echoing the declared aim of Cantoral's socio-epistemology (Cantoral, 2013), which they did not know about.

They also pointed out that they felt that developing an affective relation to the numbers modulo 12 and their fate helped them to reconstruct their relation to mathematics.

They were amazed to discover that mathematical objects and processes can be *interpreted* in different ways. They fathomed that interpretation is indeed a key notion in both choreography and mathematics.

They also appreciated the big difference between visualizing mentally a mathematical process and bodily enacting it. They disagreed with the traditional claim that constructing mental images is all that there is in mathematics.

## Discussion and open ends

The main thrust of our paper is that mathematical processes, deterministic or stochastic, naturally generate shapes, movement and dance through embodiment. More concretely, we have seen that an enactivist and metaphorical approach to mathematics suggests choreographies related to mathematical objects. Since we deem processes more relevant than structures, among mathematical objects, we addressed the question: How can we dance mathematical processes?

We then explored how to dance stochastic processes (exemplified by simple one and two-dimensional random walks) and deterministic processes (exemplified by arithmetical dynamical systems in the finite cyclic universe of the integers modulo *m*).

We implemented some pilot workshops, first with in-service primary school teachers and more recently with dancers (some of them also teachers), where the participants enacted:

- The 3-jump random walk of a frog on a row of stones in a pond,

- The arithmetical dynamical system generated by repeatedly multiplying by 3 in the universe of the integers modulo 12, seen as the usual clock.

The participants were amazed to discover that mathematical objects and processes could be bodily *interpreted* in sundry ways. In the 2022 workshop, the dancers freely interpreted the transition from one hour of the clock to another (i.e., from one integer modulo 12 to another, or from one vertex of the regular dodecagon to another). In fact, the whole choreography and bodily enactment can be seen as an "interpretation" of a mathematical process. Both in mathematics and in choreography, manifold interpretations are possible. In the case of mathematics, this is not so apparent in traditional teaching, where learning by rote and robotic procedures are emphasized.

The added liberty involved in the interpretative enaction of each dancer of an abstract dynamical system is not superfluous or irrelevant; it might seem so from a purely mathematical viewpoint, but it is not so for the learners, especially if we do not see mathematics as a corpus of knowledge carved on marble but as a shared human experience.

We observed that bodily enacting abstract mathematical notions is indeed possible and makes a huge difference for the learners, as the participants in our last pilot workshop reported.

We see the emergence of non-hierarchical procedures among the dancers as an avatar of human "swarm intelligence," especially in the case of collaborative work or collective improvisation; indeed, dance lends itself better to experience the collective than mathematics as usually taught or practised. We can see dance as a game in space, time, and movement, unfolding in the logical structure of a choreography. Working in the collective is crucial in this sort of game; each one does something on his/her own and all together create an ensemble, relating to one another (a sort of "egregor," some would say). So, dance creates a relationship with the collective.

In traditional mathematics, on the contrary, learning appears as a "one-person show" where learners are silent and isolated in the classroom, evaluated verbally and symbolically in isolation, and peer communication and collaboration are forbidden so as "to separate the wheat from the chaff" efficiently. Criticism of traditional repressive and selective mathematical teaching has been present, however, in mathematics education since the 1960s; see, for example, Brousseau (1965), where playful suggestions for "mathematics lessons without words" at the early primary school are presented. The role of dance in relation to the non-verbal and embodied learning of mathematics was not yet duly appreciated at the time. For a recent criticism of cognitive abuse in mathematics teaching, see Watson (2021).

When mathematics intertwines with dance, however, the crucial role of the collective can "percolate" from dance to mathematics, so to say. It becomes easier, then, for a group of learners to engage in a *collective improvisation* (a cognitive random walk) when tackling a mathematical problem collaboratively.

We also remark that in dance, we experience and observe; each dancer has a (mental and bodily) memory of what she and the others did before, how their trajectories crossed, for instance. Interestingly, even if one of our arithmetical flows is not (globally) reversible, each dancer can reverse their own trajectory.

As open ends, questions, and conjectures, we may mention the following.

Each mathematical process suggests a broad spectrum of possible choreographies.

Different metaphorizations of a given mathematical process generate different choreographies. We could explore "metaphor generated choreographies."

Could collective choreographic improvization trigger new mathematical ideas, as sheer collective mathematical improvization does?

Our experimentation suggests that the notion of visualization should be revisited, not as just a mental mechanism but as a whole-body activity.

An open question among others is how, besides enacting spatial patterns on the floor, could the dancers enact the flow of motion in their own bodies.

Regarding limitations of our study, it would be interesting to try our approach to intertwining dance and mathematics with a broader spectrum of learners, both mathematically inclined and not. Up to now, we have implemented it just with in-service primary school teachers and a small number of professional and amateur dancers interested in exploring the relationship between dance and mathematics.

Also, it may seem the mathematics involved in our mathematical processes are too simple. In fact, the mathematics underlying *multiplicative* dynamical systems are subtler than those underlying *additive* dynamical systems (where we just *repeatedly add* a fixed number k). In our case, the phenomenology is richer, and the dancers/learners have the possibility of discovering, by themselves, for instance, that the fact that 13 is the *multiplicative inverse* of 5 modulo 16 "means" that the associated flows, to $k$ = 5 and $k$ =13, in the integers modulo 16, are one the *reverse* of the other. Also, the fact that under multiplication by $k$ = 12, we have four numbers, which go to the "attractors" 4, 8, and 12 modulo 16 (so the flow is not reversible) has an arithmetical "explanation": in each case, those four numbers differ by multiples of 4 (like 7, 11, 15) and 12 x 4 = 48 = 16 x 3 = 0 modulo 16. Nevertheless, although this flow is not reversible, each dancer could reverse their path, relying on their body memory.

On the contrary, all flows associated with non-zero $k$'s are reversible in the integers modulo 17 and, more generally, in the integers modulo $m$ for all prime $m$. Recall in this context that the reversibility of a process is a key question in systems theory, which we are approaching here in an elementary and embodied way.

As a final caveat, to say that our approach to mathematics and dance represents mathematical facts in an iconic, embodied, enactive way is perhaps not the whole story and a somewhat biased description. We might better say that "something hard to fathom" has a symbolic mathematical facet, an iconic facet, and an enactive facet, each being a metaphor for the other. This "something" is being somehow constructed by us, enactively in the sense of Varela et al. (1991).

## Acknowledgements

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## Section 6

# MATHEMATICS & SCIENCE & TECHNOLOGY & COMPUTATIONAL THINKING

# Interdisciplinarity of science, mathematics, and technology in a context of elementary school scientific investigation using a virtual manipulative

Takam Djambong[1]

*This study is qualitative research carried out with 29 Grade 7 and 8 primary school students. The objective of this study was to bring out, in a context of scientific investigation on the principle of buoyancy with the help of a virtual manipulative, the representations, thinking processes, and learning approaches mobilized by the pupils, with a view to constructing a conceptual scheme of the components of an interdisciplinary approach involving science, technology, and mathematics. The conceptual scheme of the components of STEM interdisciplinarity that emerged from the data analysis is based on three pillars, which are the learning, epistemological, and cognitive components and articulates in a complex way the learning approaches, the different modes of symbolic representations, and the thinking processes resulting from the preceding components.*

Keywords: *STEM education, interdisciplinarity, symbolic representation, virtual manipulatives, buoyancy.*

## Context, background, and objectives

Today's elementary science education evolves as students are confronted with the analysis or more complex problem-solving through the integration of knowledge, know-how, and interpersonal skills resulting from interaction of different disciplines (science, mathematics, and technology in particular). Complex problem-solving processes increasingly involve the manipulation of computational models (virtual manipulatives) of scientific phenomena (Weintrop et al., 2016). Part of the projects carried out within the CompéTICA network between 2015 and 2018 conducted several case studies of innovative practices (Freiman et al., 2017).

So far, the case studies conducted in the CompéTICA network have explored various aspects of skill development in different life contexts (social, academic, personal, professional). These studies have tried to highlight some best practices that allow the articulation of techno-instrumental skills and 21st century skills. However, the place of STEM interdisciplinarity in the continuum of digital skills that combines techno-instrumental skills with soft skills remains to be clarified. The role that STEM interdisciplinarity could play in the junction between techno-instrumental skills, soft skills, and disciplinary skills in a school context remains to be elucidated.

This article presents one of these studies, which aimed to: (1) identify the different modes of symbolic representation mobilized by elementary school students from an interdisciplinary perspective; (2) bring out the different thinking processes used by the students during the scientific investigation activity on Archimedes' principle using a virtual manipulative; (3) bring out a conceptual scheme to explain the articulation between the mobilization of different modes of symbolic representation, the

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mobilization of different thinking processes and the interdisciplinarity of science, mathematics and technology.

## The conceptual framework

## Integrated STEM education

The main interest of our study is that it places the activity of learning science using a computational artifact (the virtual manipulative) within the conceptual framework of the integral approach described by Kelley and Knowles (2016). This approach is conveyed by the STEM movement in connection with the emergence of new forms of 21st century literacy of which learning integration through transdisciplinary, multidisciplinary, or interdisciplinary approaches seems to be one of the main cornerstones today. This third interest of our study raises, in parallel, the need to build a didactics of STEM disciplines to frame this movement, which is emerging as one of the important axes of 21st century learning in the digital era.

Sengupta et al. (2018) pointed out that the use of computational modes of representation of data, information, concepts, processes, or phenomena in the context of learning STEM subjects, implies that students are confronted with the manipulation of multiple forms or modes of symbolic representation of knowledge. The simultaneous use of multiple forms or modes of symbolic representation of knowledge goes beyond the traditional operations of coding or programming, which can themselves, from our point of view, be seen as alternative modalities of data or information representation.

STEM education can link scientific inquiry, by formulating questions answered through investigation to inform the student before they engage in the engineering design process to solve complex problems (Kennedy & Odell, 2014). The investigative approach in science (inquiry process) is a favorable context for the emergence of pedagogical practices aimed at integrating knowledge through interdisciplinary learning activities (Hasni et al., 2015; Kelley & Knowles, 2016; Walker et al., 2018). On the other hand, the potential of computational modelling to improve students' understanding of science and mathematics has been documented by several authors (Repenning et al., 2010; Wilensky, 2014). For Wing (2014), computational thinking is the driving force of scientific research that has potential for cognitive development and conceptual change in students when they are engaged in relational thinking through the construction of links between the scientific method (Landriscina, 2013) and mathematical thinking (Savard et al., 2013; Beaufils, 2000). Computational thinking highlights the advantages of involving students in the manipulation of computational models of scientific concepts and phenomena. This allows students to articulate and match different registers of symbolic representation to examine different facets of the same phenomenon. This was supported by Gauthier (2014) with students building and exploring models using dynamic algebra software.

## Interdisciplinary learning approaches

Sriraman et al. (2008) have shown that pedagogical practices that build on an interdisciplinary approach by emphasizing the connections and interdependence that can exist between different disciplines (mathematics, biology, chemistry, and physics), in a meaningful, authentic, and situated learning context, can have a positive impact on increasing students' personal interest in science and mathematics.

The integration of STEM disciplines aims to place the student at the heart of a process of meaning-making involving a complex set of thinking processes, learning approaches and symbolic representation systems. This integrative vision of knowledge could be supported by the model for restructuring the Quebec primary school curriculum in an interdisciplinary perspective (Lenoir, 2020). This interdisciplinary model envisages a categorization of school subjects according to their relationships to the underlying mode of representation of reality inherent in each school discipline.

Thus, in a resolutely epistemological perspective, Lenoir (2020) considers:

- disciplines that contribute to the construction and structuring of natural, human, and social reality (arts, natural, human or social sciences);
- disciplines that enable this reality to be expressed using a specific symbolic representation system (mathematics, languages);
- disciplines that help to establish relationships with reality (physical, moral, or religious education, and technology).

Moreover, the interdisciplinary perspective presented in Lenoir's model is designed for action and allows for a close link between the cognitive dimensions and the instrumental and procedural dimensions of learning.

## Virtual manipulatives and simulations

The construction of knowledge can be done through the manipulation of digital tools (virtual manipulatives, programming, simulation or game design environments, creative labs, robotics, etc.) and lead to the construction by students of symbolic artefacts (models and conceptual representations) or tangible artefacts (physical models), as evidence of the articulation between abstract and procedural thinking (Papert, 1991).

According to Moyer-Packenham and Bolyard (2016, p. 3), a virtual manipulative can be defined as "*an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge."* According to these authors, the term "interactive" was used to distinguish dynamic computational representation enabling interactions with the users and simple static images. The term "technology-enabled" refers to the fact that all virtual manipulatives are considered, not only the Web-based ones. Finally, the concept of "visual representation" emphasizes the fact that a mathematical idea could be represented by a pictorial image.

The use of virtual simulations also makes it possible to construct learning tasks that encourage students to mobilize thinking processes or cognitive skills related to several disciplines. Thus, the integration of the use of virtual simulations in teaching would not only promote the development of transdisciplinary skills in learners, but also their ability to mobilize knowledge and know-how from several disciplines to carry out a learning task.

In addition to the contributions already mentioned above, simulations in certain virtual environments would not only make learning attractive and meaningful but would also increase students' motivation and interest in learning the subject concerned. Such simulations would also allow the development of in-depth learning that could lead to the conceptual understanding of the phenomena studied (Freiman et al., 2012).

The concept of simulation could be defined as:

- a dynamic and interactive representation of a model, an event, a device or a phenomenon constructed to present and understand how a system works (Depover et al., 2007; Legendre, 1993);
- an intermediate layer (simulation environment) that favours a relationship between events and phenomena in the real world (or material world) and the interpretation of scientists in terms of theories, principles, laws, or models (Beaufils & Richoux, 2003);
- an interactive representation of a system under study based on a model (simplified representation of reality) of that system (Landriscina, 2013). This definition of simulation, however, limits its meaning to situations specific to educational and scientific contexts;
- a tool with cognitive potential that can foster the development of high-level cognitive skills, such as creative thinking, abstract thinking, complex problem solving, and the ability to work in teams (Jonassen & Carr, 2000).

Beaufils and Richoux (2003), have highlighted the didactic potential of simulations in terms of possible relationships and connections with experimentation, study and manipulation of models, effects on conceptual change and cognitive development, or in terms of links with the implementation of a multi- or interdisciplinary approach in science learning (Landriscina, 2013).

## Research question

This study was driven by the following research question: *Which conceptual scheme of interdisciplinary connections emerge between STM disciplines in a context of scientific inquiry on buoyancy using a virtual manipulative environment?*

## Methodology

This study is a qualitative and interpretative case study informed by Merriam (1998) inductive approach. Our qualitative research is based on the interpretative paradigm and on a constructivist epistemology, which postulates the existence of multiple realities (Guba & Lincoln, 1994)

Data was collected from 29 students (11 from Grades 7, and 18 from grade 8) in two Francophone schools in Moncton (New Brunswick). The data collection techniques consisted of a concomitant think-aloud protocol, retrospective semi-structured interviews, and written and digital traces of students' work. A learning scenario with three tasks was designed according to the Predict-Observe-Explain strategy (Gauthier, 2014). Interview guides and a logbook were used as data collection instruments. The data analysis strategy was based on conceptualizing category analysis according to the coding approach leading to data reduction (Thomas, 2006). The data analysis was carried out in four steps:

- Microanalysis (initial coding of the data corpus)
- Categorization (open coding)
- Articulation of categories (axial coding)
- Category integration (selective coding)

In our research, think-aloud protocols and activity traces were collected from students involved in the three science problem-solving tasks using a virtual manipulative from PhET simulations website[2]. These tasks were designed to be implemented using the Predict-Observe-Explain strategy. The three problem-solving tasks proposed to students in our experiment were designed based on specific concepts (*Archimedes' Principle, buoyancy, and density*) taught in the grade 7 and 8 science and technology curriculum.

In the first task, students were asked to explain and justify why objects float, sink, or remain indifferent in certain liquids, while in the second task, they were asked to identify the factors on which buoyancy depends. In the third task, the students had to characterize the force responsible for the buoyancy principle through a mathematical modelling approach. The three tasks designed within the framework of the pedagogical scenario, which was proposed to the students, present a didactical, pedagogical, and techno-instrumental interest.

From the didactical point of view, the learning theme on the buoyancy principle is a classic case showing the extent to which students at different levels (primary and secondary) can maintain, over time, all sorts of inappropriate prior conceptions (Potvin, 2011), which nevertheless manage to survive even with the most convincing pedagogical lectures of teachers from the point of view of the explanatory power of the phenomenon. The goal of the different tasks that were proposed in the prediction phase was to bring out these initial conceptions or representations of the students as empirical data to be analyzed.

From a pedagogical point of view, in addition to achieving the learning outcomes (general, transdisciplinary, and specific) prescribed by the grades 6-8 science and technology curriculum, the pedagogical scenario was designed to promote collaboration between students through the interactions generated by the socio-cognitive conflict induced by the teamwork designed for the students. On the other hand, the nature of the proposed tasks was to encourage interdisciplinary approaches, requiring the mobilization, by the students, of concepts or practices from several disciplinary fields (science, mathematics, and technology).

On the techno-instrumental level, the pedagogical scenario proposed to the students involved the use of a virtual manipulative on the Archimedes principle as a tool for scientific investigation. The virtual manipulative that was used from the "PhET simulations" suite and was developed under an open-source license by a research team from the University of Colorado in the United States, also has the particularity, according to some authors, of promoting the implementation of a modelling approach in science (Shen et al., 2014).

## Procedure

Table 1 below shows schematically the four stages and the timeline of the methodological design that we implemented after having chosen the locations of schools and selected the two grades and the 29 students who took part in the experimentation.

[2] PHET Interactive Simulations. (n.d.). *Interactive Simulations for Science and Math*. https://phet.colorado.edu/

**Table 1:** ***Timeline of the methodological procedure***

| Stage | Activities carried out | Raw data analyzed | Analysis strategy implemented | Results | Timeline |
|---|---|---|---|---|---|
| **Deep preliminary reading of a rich corpus segment and initial coding** | *-First phase of data collection and transcription*<br>*-Careful reading of the corpus*<br>*-Labeling (Annotation of the corpus)*<br>*-Open coding (creation of codes)* | *-Various research journal entries*<br>*-observation notes*<br>*-Verbatim of corpus (interviews and thinking protocols) or transcribed student activity records* | *Line by line analysis* | *-Immersion in the research environment*<br>*-Discovery of the research context*<br>*-Initial coding of the corpus* | *May 2017* |
| **Categorization** | *-Theoretical sampling*<br>*-Second phase of data collection and transcription*<br>*-Open coding* | *-Verbatim of corpus (interviews and thinking protocols) or transcribed student activity records*<br>*-Various research journal entries*<br>*-Observation notes* | *Analysis by conceptual categories* | *-Identification of the main conceptual categories, sub-categories and central categories, phenomenon studied (Categorization)*<br>*-10 conceptual categories created* | *June-September*<br>*2017* |
| **Articulation of categories** | *-Axial coding* | *-Verbatim of corpus (interviews and thinking protocols) or transcribed student activity records*<br>*-Various research journal entries*<br>*-Observation notes* | *Analysis by conceptual categories* | *-Characterization and pairwise articulation of conceptual categories that emerged from the initial and open coding phases*<br>*-Creation of 3 central categories* | *October - November*<br>*2017* |
| **Integration of core categories in a conceptual scheme** | *- Selective coding* | *-Verbatim of corpus (interviews and thinking protocols) or transcribed student activity records*<br>*-Various research journal entries*<br>*-Observation notes* | *Analysis by conceptual categories* | *-Creation of a conceptual diagram integrating the main core categories identified* | *December 2017-February 2018* |
| **Final report writing** | | | | | *March 2018 – July 2018* |

## Coding process main steps

In this study, we implemented a six-step coding process inspired by Thomas' (2006) model and summarized in Table 2 below.

**Table 2: Main phases of the analysis according to the process of coding the raw data corpus leading to data reduction**

| Step 1 | Step 2 | Step 3 | Step 4 | Step 5 | Step 6 |
|---|---|---|---|---|---|
| | | | | | |
| Prepare raw data. | Preliminary, careful, and thorough reading of raw data. | Identification of text segments specifically related to research objectives and questions. | Coding of identified text segments to create categories. | Revision and refinement of categories by reduction of redundant or similar categories. | Design of a conceptual scheme that integrates the key categories identified or developed. |
| | | | | | |
| Prepare and save all raw data in a common format. | Read and summarize the data in order to become familiar with the content. | Select several segments of text having a specific and unique meaning for their codification. | Label text segments to create multiple initial categories. | Retain the key categories to integrate into a conceptual scheme. | Propose a conceptual scheme of the phenomenon or process studied. |
| | | | | | |

# Results

## Interdisciplinarity components, core categories and conceptualizing categories

We begin by presenting in Table 3 below the main components, core categories, and conceptualizing categories that emerged from our study and on which interdisciplinary learning approaches involving mathematics, science, and technology (digital learning tools such as simulations) could be based on.

Table 1 above shows that the science-mathematics-technology interaction has, in the framework of our study, a certain resonance with the interdisciplinary approaches or interactions implemented in relation to the learning process, the symbolic representation and thinking process used by the students during the problem-solving activity in science using the virtual manipulative on the Archimedes principle. The three central categories were essentially intended to show how the mobilization of cognitive skills, learning approaches or symbolic representations specific to the three disciplinary fields of science, mathematics, and computer science surface in the students' discourses. These interactions thus highlight the possible interdisciplinary links that the pupils establish more or less consciously during their learning activity. The mobilization of these skills set by the students relies upon the nature of the tasks proposed and the prescriptions of the learning scenario, which induce the use of language, concepts or approaches specific to different disciplinary fields (science, mathematics, and technology literacy). Tables 4, 5 and 6 below present excerpts supporting this evidence.

**Table 3: Interdisciplinarity components, core categories and conceptualizing categories**

| Interdisciplinarity component | Core categories | Conceptualizing categories |
|---|---|---|
| **Learning component** | **Learning process** | Collecting and analyzing data |
| | | Identifying patterns in quantitative data |
| | | Manipulating affordances for operating on data |
| **Epistemological component** | **Symbolic representation process** | Building a symbolic representation of a phenomena |
| | | Building a mathematical representation of a phenomena |
| | | Manipulating a computational representation of a phenomena |
| **Cognitive component** | **Thinking process** | Modifying an initial representation of a phenomena |
| | | Mobilizing cognitive skills related to science |
| | | Mobilizing cognitive skills related to mathematics |
| | | Mobilizing cognitive skills related to technology literacy |

## Learning component

Excerpt 28 in the Table 4 below, for example, highlights certain aspects of science-mathematics-technology interdisciplinarity. The students (e.g., S28, S29 and S30) mobilized the concepts relating to science (gravity and density) and mathematics (numbers) by implementing a certain number of didactic approaches such as using mathematical symbolic representation in a process of constructing representations in science (X = 9.8*V*density of the liquid, for example). Excerpt 28 also shows all the numerical operations carried out by students S28, S29 and S30 following the manipulation of the affordances of the virtual manipulative in a context of scientific investigation. An aspect of interdisciplinarity highlighted by their conceptual model therefore refers to the possibility given to students to mobilize and combine different didactic approaches and techniques for representing knowledge specific to different disciplinary fields (STEM for example) as evidenced by the following statement:

**Table 4: Learning component**

<table>
<tr><td rowspan="1">Investigation process<br>(Science)</td><td colspan="2">S20: Ok uhhh for the wood...the brick, the volume of the brick...<br>S21: We have to estimate the volume of the brick?<br>S20: We have to estimate the volume of the brick. How are we going to put the width of the wood?<br>S21: Ah...OK. If the mass of the wood is 4 kg, will it sink or float?<br>(Excerpt 77: Verbatim PP- Task-1-Predict)</td></tr>
<tr><td rowspan="2">Modeling process (Maths)</td><td colspan="2">S8: "X, Y and Z all have the same units, tenths and hundredths are not the same...".<br>(Excerpt 31: Written traces TE-Task-3-Explain-and-Model)</td></tr>
<tr><td colspan="2">S5: "X, Y and Z are almost all the same.<br>(Exceprt 32: Written traces TE-Task-3-Explain-and-Model)<br>"S30: We would put gravity<br>S28: Calculate and put the difference of... You have to calculate the difference between the two volumes<br>S29: What are you doing? Do the 107.50 with...your calculator...<br>S30: Do them with my computer?<br>S29: Check<br>S28: 107.50 - 100.<br>S29: So it's 7.50<br>S28: Calculate the value of the quantity X =9.8*V* density of the liquid... It's true that you can read that like you're better than me there..."<br>(Excerpt 28: Verbatim PP-Task-3-Simulate-and-Observe)</td></tr>
<tr><td rowspan="2">Instrumental genesis process (Technology literacy)</td><td>S5: "I was not used to using the software, then uh.... I wasn't used to kilograms, then density...".<br>(Excerpt 1: Verbatim Interview of student Bristan)</td><td rowspan="2"></td></tr>
<tr><td>S5: "Yes in oil. I click on thrust.... I'll click on gravity...same volume...same density...it still changes...and both float. And in water...if I put the same mass."<br>(Excerpt 2: PP Verbatim – virtual manipulative handover)</td></tr>
</table>

## Cognitive component

The excerpts in Table 6 below show that the students were able to articulate elements of scientific thinking through the construction of a mental model or conceptual representation of the principle of buoyancy (excerpt 81), mathematical thinking through the pattern recognition in a sequence of data (excerpt 82), and algorithmic thinking through the implementation of the if condition...then... loop (excerpt 104)

**Table 5: Cognitive component**

<table>
<tr><td rowspan="2">Scientific thinking (building a mental model or conceptual representation of buoyancy principle)</td><td>S5: If the kind of material is heavier than the density of the water, the body will sink, but if they are exact (equal), it doesn't matter, and then if it is less heavy, it will often float.<br><br>(Excerpt 81: Verbatim interview of student Bristan)</td></tr>
<tr><td>S6: If the density of the two things are the same, it won't matter, if the density of the liquid is smaller than the density of the object, the object will sink, and if the density of the liquid is bigger than the density of the object, it will float...<br><br>(Excerpt 82: Verbatim interview of student Juliana)</td></tr>
<tr><td>Mathematical thinking (Identifying patterns in a sequence of data)</td><td>
<table>
<tr><td rowspan="2">Valeurs à mesurer</td><td colspan="2">Solides plongés dans l'hélium liquide de masse volumique 0,15 Kg/L</td><td colspan="2">Solides plongés dans l'eau de masse volumique 1 Kg/L</td></tr>
<tr><td>3 kg de bois</td><td>3 Kg de glace</td><td>3 Kg de brique</td><td>3 Kg d'aluminium</td></tr>
<tr><td>Volume $V_1$</td><td>100,00</td><td>100.00</td><td>100.00</td><td>100.00</td></tr>
<tr><td>Volume $V_2$</td><td>107,50</td><td>103.76</td><td>101,50</td><td>101,11</td></tr>
<tr><td>Différence $V = V_2 - V_1$</td><td>7,50</td><td>[illegible].76</td><td>1,5</td><td>1,11</td></tr>
<tr><td>Grandeur X = 9,8*V*masse volumique du liquide</td><td>11.03</td><td>4.79</td><td>14.70</td><td>10.88</td></tr>
</table>
</td></tr>
<tr><td rowspan="2">Algorithmic thinking (through the instrumental use of the virtual manipulative)</td><td>"If the densities are equivalent, the object will be indifferent. If the density of the liquid is smaller than that of the object, the object will sink. If the density of the liquid is greater, the object will float''<br><br>(Excerpt 104: Student Handouts S1 and S2 TE-Task-1-Explain-and-Model)</td></tr>
<tr><td>S24: ''If it sinks it's because the density is greater than the liquid. Euuhhh...If it floats, it's because the density is smaller than the liquid and if it stays indifferent, it's because the density is equal to the liquid".<br><br>(Excerpt 105: Verbatim PP-Task-1-Simulate-and-Observe)</td></tr>
</table>

## Epistemological component

The epistemological component of this study highlights the construction, expression, manipulation, and modification of symbolic representations (as a means of expressing knowledge) by students involved in a scientific investigation process.

The excerpts show that the students, during this learning activity, were able to:

- build mental models of the concept of buoyancy (excerpt 45),
- develop a mathematical model of the principle of buoyancy (excerpt 29),
- manipulate the computer model of the buoyancy principle using a virtual manipulative (excerpt 2),
- evolve their initial representations of the principle of buoyancy through the process of conceptual change (excerpt 56).

**Table 6: Epistemological component**

<table>
<tr><td>Symbolic representation construction<br>(Science)</td><td>S3: Euuuh... it's that if the density of the object is smaller than the density of the liquid, erm... it would float, If you want it to float, you need the density of the liquid to be greater than the density of your object. If you want it to sink, it has to be the same amount of density of the solid and a small amount of density of the liquid.<br><br>(Excerpt 45: Verbatim PP-task-1-Explain-and-model)</td></tr>
<tr><td>Symbolic representation expression<br>(Mathematics)</td><td><table>
<tr><td>Grandeur X = 9,8*V*masse volumique du liquide</td><td>11.03</td><td>4.79</td><td>14.7</td><td>10.89</td></tr>
<tr><td>Valeur du poids P</td><td>29.40 N</td><td>29.40N</td><td>29.40N</td><td>29.40</td></tr>
<tr><td>Valeur du poids apparent P<sub>a</sub></td><td>18.38 N</td><td>24.60N</td><td>14.70N</td><td>18.51</td></tr>
<tr><td>Comparer P et P<sub>a</sub></td><td>P>Pa</td><td>P>Pa</td><td>P>Pa</td><td>P>Pa</td></tr>
<tr><td>Valeur de la grandeur Y = P – Pa</td><td>11.02</td><td>4.8</td><td>14.7</td><td>10.89</td></tr>
<tr><td>Valeur de la poussée Z affichée par le logiciel</td><td>11.03N</td><td>4.80N</td><td>14.70N</td><td>10.89N</td></tr>
</table>Extrait 29 : Traces écrites des élèves S28, S29 et S30 TE-Tâche-3-Simuler-et-oberver</td></tr>
<tr><td>Symbolic representation manipulation<br>(Technology literacy)</td><td>"Yes in oil. I click on thrust.... I'll click on gravity...same volume...same density...it still changes...and both float. And in water...if I put the same mass."<br><br>(Excerpt 2: PP Verbatim - Handover)</td></tr>
<tr><td>Symbolic representation Modification<br>(Conceptual change)</td><td>"I learned a lot of things like with different solids and different materials, there were different things that could like sink...float, like be indifferent. Like there were a lot of things that I didn't know too...".<br><br>(Excerpt 56: Verbatim interview with student Odette)</td></tr>
</table>

The third guiding principle advocated by the Science and Technology 6-8 curriculum and implemented during this learning activity is interdisciplinarity as a lever likely to promote the integration of knowledge and approaches specific to different disciplinary fields with a view to solving an identified problem. Thus, the educational scenario offered to the students enabled them to mobilize knowledge related not only to science, but also to mathematics and technology literacy (in its techno-instrumental dimension). Interdisciplinarity has thus resulted in the fact that students have been able to use intentionally and consciously mathematical concepts or operations in the context of problem-solving in science. The data indicate that didactical, epistemological, and thinking process components should also be taken into consideration when it comes to building interdisciplinary-based learning activities involving science, technology, and mathematics.

The three central categories that emerged from this study highlight a certain resonance with the interdisciplinary interactions, which were observed in relation to the modes of thinking and in turn mobilized, the didactic approaches as well as the type of symbolic language used and manipulated by the students during the problem-solving activity in science using the virtual manipulative on Archimedes' principle. These central categories, in conjunction with the conceptualizing categories that follow (see table 3), have made it possible to develop a conceptual scheme of a science-mathematics-technology interdisciplinarity presented in the next section.

## Our conceptual scheme of STEM interdisciplinarity components

The conceptual scheme in Figure 1 below is based on the core and conceptual categories that emerged from our research question. This diagram highlights the main components on which interdisciplinary learning approaches could eventually be built based on a complex articulation between cognitive, epistemological, and didactic components from the different contributing disciplines (Science, Mathematics, and Technology literacy).

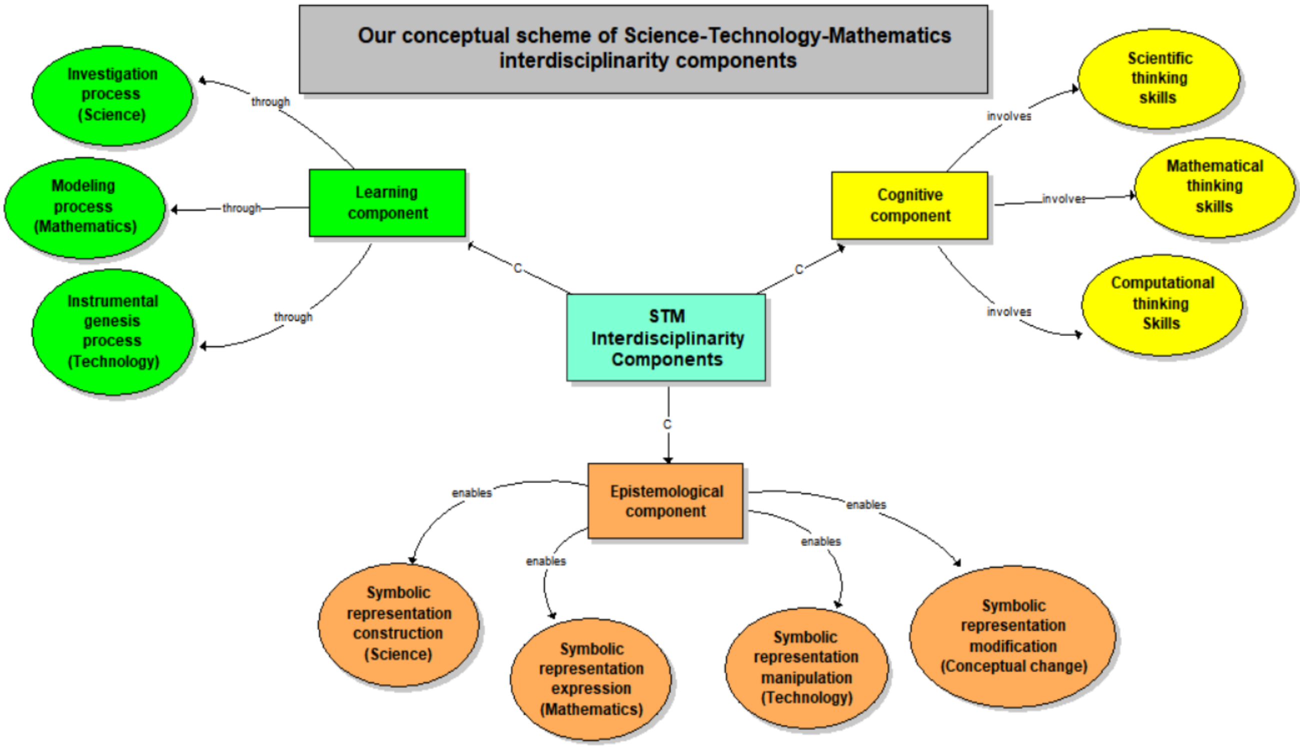


**Figure 1: Conceptual scheme of Science-Technology-Mathematics interdisciplinarity components**

The analysis of empirical data carried out in this study led to a conceptual scheme of interdisciplinarity components. This conceptual scheme indicates that the students' activity was essentially based on a process of constructing, manipulating, expressing, and modifying their representations of Archimedes' principle (*epistemological component*), which was itself implemented by relying on the scientific investigation process coupled with a modelling process and instrumental genesis process (*learning component*). The specific context in which the learning activity took place favored the mobilization by students of scientific thinking skills, mathematical thinking skills, and technology literacy skills (*cognitive component*).

Regarding the first objective, the students in a context of scientific investigation of buoyancy (*Archimedes principle*) using a virtual manipulative had to articulate the development of a mathematical model of the Archimedes principle, the manipulation affordances of the computational model of Archimedes' principle, as well as the verbal modelling of buoyancy in order to analyze the phenomenon explored.

Regarding the second objective, the data collected during the study shows that the students used mathematical thinking, some traces of computational thinking, and the scientific method process in exploring buoyancy.

Finally, considering the third objective, the data suggests some disciplinary interactions between science, mathematics, and technology concepts, as a potential vector for the science, technology, and mathematics (STM) knowledge integration in a context of science learning in elementary school. These STM interactions are mainly represented by the ability of students to handle and articulate different modes of symbolic representations and different thinking processes to analyze and apprehend a scientific concept (the principle of Archimedes) under various perspectives.

## Scholarly significance

The context of science investigation in elementary school using a virtual manipulative seems to have created favorable conditions aimed at:

- bringing out several modes (conceptual, mathematical, and computational) of symbolic representations of the phenomenon studied;
- bringing out several learning approaches (scientific investigation, mathematical modeling, and instrumental genesis of usage patterns);
- encouraging students to mobilize skills or concepts specific to different subject areas (science – mathematics – technology).

The conceptual scheme of the STEM interdisciplinarity components that emerged from the data analysis is based on three pillars of learning, epistemological, and cognitive components. This conceptual scheme articulates in a complex way the learning approaches, the various modes of knowledge representations, and the thinking processes resulting from the previous components.

Firstly, the learning activity with the virtual manipulative thus allowed students to implement to some degree certain aspects of STEM interdisciplinarity, especially in terms of mobilizing concepts, and approaches or ways of representing knowledge from these different disciplinary fields.

Secondly, the learning activity using the virtual manipulative also aimed at achieving certain transdisciplinary learning outcomes (TLOs) prescribed by the 6-8 science and technology curriculum such as critical thinking and the pedagogical use of technology-enhanced learning environments.

Finally, our results help to highlight the importance of technologically rich learning environments in a context of scientific inquiry as a means for the integration of science, technology, and mathematics in an interdisciplinary perspective in elementary school. They also bring out the importance of bringing together different types of modelling (mathematical, computer, and verbal) in an interdisciplinary perspective of integrating knowledge from science, mathematics, and technology in elementary school.

## Conclusion

In conclusion, this study shows that there could be a link between the students' ability to solve the tasks proposed, the type of targeted skills related to computational thinking, and the degree of difficulty or complexity of the proposed tasks. The influence of the programming environment to which the students were exposed in the context of problem-solving tasks during the intervention, is difficult to demonstrate given the limitations associated with the experiment (small size of sample, non-randomized sample, lack of a control group). However, this study justifies the need for further studies to establish the validation of the proposed tasks based on more solid empirical evidence. It could thus be useful to look at the effect that the nature of the pedagogical intervention in programming environments (visual versus tangible) could have on the validation of the proposed set of tasks. For this purpose, more subtle research design for the study and instruction for the problem-solving tasks are needed. Our study shows that interdisciplinary approaches could go beyond the simple integration of knowledge and skills from different disciplines. It would also be relevant to take into consideration the didactic approaches, the epistemological frameworks and the thinking processes underlying each discipline in order to build learning activities in an interdisciplinary perspective.

## Acknowledgement

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# Characterizing problem handling in the intersection between computational thinking and mathematics

Raimundo Elicer[1] and Andreas Lindenskov Tamborg[1]

*This communication reports on interventions in collaboration with a mathematics teacher aimed at developing and implementing teaching materials that introduce students to problem-solving tasks that integrate computational thinking (CT) and mathematics. Drawing on classroom observations, we seek to characterize the notion of CT-driven problem handling in the mathematics classroom. In this intersection, problem handling or solving has a prominent place, though its meaning may not be the same. We delineate three abilities in this characterization: to solve computational and mathematical problems through effective modeling, to pose own real-life inquiries as computational and mathematical problems, and to judge which elements of the solution strategies can be transferable.*

*Keywords: Problem solving, computational thinking, mathematical competencies.*

## Introduction

Computational thinking (CT) has gained relevance in the educational research community since Wing (2006) defined this theoretical construct as a set of trainable skills and abstract ways of thinking. As a consequence, several countries are including programming and aspects of CT in their school curricula (Bocconi et al., 2022). In many cases, due to more or less apparent synergies, it is the mathematics syllabi and teaching practices that have adopted the responsibility to include CT. This paper is focused on one inarguable common element between CT and mathematical competencies, namely, the handling of problems.

The ability to solve problems is at the core of many definitions and characterizations of CT. Wing portrays CT as “a way humans, not computers, solve problems” (Wing, 2006, p. 35), and argues that it encompasses “the thought processes involved in formulating a problem and expressing its solution(s) in such a way that a computer—human or machine—can effectively carry out” Wing (2017, p. 2). A systematic literature review defines CT more comprehensively as “the conceptual foundation required to solve problems effectively and efficiently (i.e., algorithmically, with or without the assistance of computers) with solutions that are reusable in different contexts” (Shute et al., 2017, p. 142). Mathematical situations are certainly among this multiplicity of contexts, as understanding and making sense of computations is an integral part of mathematics (Li et al., 2020).

The literature validates problem solving as an ultimate purpose of CT in mathematics learning. For example, both Weintrop et al.’s (2016) and Pérez’s (2018) characterizations of CT in mathematics classrooms frame computational problem-solving as an essential category of practices. Kallia et al.

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(2021, p. 20) define CT in mathematics education as “a structured problem-solving approach in which one is able to solve and/or transfer the solution of a mathematical problem to other people or a machine.” Overall, CT-related practices and abstract ways of thinking can be a tool to aid mathematical problem solving.

There is a vast tradition of problem solving (PS) in mathematics education, and it can be traced back to George Pólya. As a mathematician and educator, he devised a generalized strategy for PS (Pólya, 1957) consisting of understanding the problem, devising a plan, carrying out the plan and looking back. Later, Schoenfeld (1985) built explicitly on Pólya to describe four categories of mathematical behavior centered around problem solving: resources, heuristics, control and belief systems. This time, backed by empirical evidence in learning settings. Schoenfeld (1992) argued that Polya’s influence resides in framing of PS as a fundamental element of mathematical invention, contrary to the Euclidian-style deductive character of how it is presented. This agenda has influenced the goals and ways of learning and teaching mathematics ever since.

In general, problem solving has served as a means for teaching mathematics (Cai, 2010), and thus PS competences tend not to be separated from the domain-specific objectives in the curriculum (Artigue & Blomhøj, 2013). In the Danish framework for mathematical competencies (KOM), Niss and Højgaard (2011) delineate problem handling as the ability to pose and solve different kinds of mathematical problems. In their recent work, they highlight that PS concerns mathematical problems, and tackling extra-mathematical problems belongs to the modeling competency (Niss & Højgaard, 2019). Furthermore, Geraniou and Jankvist (2019) contributed a theoretical networking of mathematical and digital competence (MDC), in which PS is also mentioned. One of the three main components of MDCs involves “being able to use digital technology reflectively in problem solving and when learning mathematics” (p. 43).

PS then appears as a central aspect of both digitalized mathematics education and CT. However, we still lack empirical insights into how mathematical problem handling could (and perhaps should) be taught and learned in CT-driven mathematics classrooms. In this paper, we report from a collaboration with a teacher aiming to develop and implement teaching resources, in which students are introduced to PS tasks that integrate CT and mathematics. Based on these experiences, we seek to answer the following research question: How can we characterize the notion of problem handling in the context of programming and CT in the mathematics classroom?

## Background

Our work is conducted in Denmark, where CT was included in the compulsory school curriculum as an experimental subject in a pilot project that ran from 2018–2022. During this period, 46 schools across Denmark implemented a new subject called technology comprehension (TC). TC consisted of four main competency areas: digital empowerment, digital design and design processes, computational thinking and technological agency. A key idea in the pilot project was to gain experiences with the TC subject via systematic evaluations and to inform a near-future national scale implementation on these insights. A part of gaining these insights consisted in trying out two different implementation strategies: implementing TC as a subject in its own right and as integrated into existing subjects, here among mathematics. Both implementation strategies involved developing a new curriculum and developing teaching resources that could support the concerned teachers in teaching the new TC content. The Danish mathematics curriculum is organized into four competence areas: mathematical competencies (Niss & Højgaard, 2019) and subject-matter areas, numbers and algebra, geometry and measures, and probability and statistics. Both strategies should address the same curriculum components. Hence, to integrate TC into existing subjects, the individual competence areas of the curriculum for TC, as a subject in its own right, were to be distributed among the subjects in which TC should be integrated. In the case of mathematics, six TC components are integrated into mathematics: digital design and design processes; modeling; programming; data, algorithms and structures; user studies and redesign; and computer systems.

In our earlier work, we observed that the new TC competence and subject-matter areas were added to the mathematics curriculum without explicitly relating it to the existing mathematical competencies and subject-matter areas (Tamborg et al., 2022). Despite the potential synergies between CT and mathematical PS, relations between these areas were thus not established at the curricular level. Two additional preliminary results are worth mentioning in this respect. First, problem handling is not invoked in any of the available resources that were developed as part of the strategy of integrating programming and CT into mathematics teaching (Elicer & Tamborg, 2023). Second, the mathematics teacher with whom we collaborated to design a task strongly emphasized the necessity for students to own (delineate and pose) the computational problem (Elicer et al., 2022). Therefore, problem handling became a focus point in the classroom intervention that followed, and it is the empirical anchor for our theoretical contribution.

## Empirical basis

Our analysis draws on the development and subsequent implementation of a mathematical problem-solving task that involved CT. We developed the task on the basis of a combination of insights from research literature and refined it through a systematic collaboration with an experienced mathematics teacher over the course of several months. This teacher also implemented the task in a 6th-grade class of 18 students. The classroom intervention consisted of three 90-minute sessions that took place on the spring of 2022. The task concerned geometry, and the students were to experience this mathematical topic and to engage in problem handling related to it in the widely known block programming environment called Scratch. The first session mainly introduced the students to Scratch, so that they were acquainted with navigating this environment. In the second session, students were asked to program different regular polygons in Scratch and explore the relation between the number of sides and the corresponding turning angles (cf. Elicer et al., 2022). In the third session, students were asked to draw a skyline of their choice by applying their insights gained from experimenting with polygons (angles and side lengths) in Scratch. In our analysis of the characteristics of problem tackling in the context of programming and CT in the mathematics classroom, we mainly draw on the tasks in sessions two and three as they were implemented by the teacher and carried out by the students in the classroom. We collected data on these matters via participant classroom interventions, which was documented by fieldnotes and collected student products (such as skylines drawn by students).

## Characterizing problem handling

Based on the empirical foundation, we identified three main traits that characterize problem handling in the context of CT-embedded mathematics education. These consist of the disciplinary nature of the problem (mathematical, computational or otherwise), the ownership of the problem (who poses it), and the transferability of solution strategies. In what follows, we briefly define each of these issues, illustrate them by selected episodes of our empirical basis, and relate them to the existing literature.

## Disciplinary nature of the problem

The first issue concerns the disciplinary nature of the problem at hand, which has several components. That is, the first question is which domain of knowledge the problem is coming from. Is it a mathematical problem solved by computational means or vice versa? In principle, our design decision was proposing a problem whose disciplinary nature is blended.

In the aforementioned task, figuring out the pattern or a general expression of the internal or external angle of any given regular polygon is a mathematical problem in the sense of Niss and Højgaard (2019). First, it is a purely geometrical problem and not an extra-mathematical one. If any modeling is involved, it is that of modeling a geometrical situation into algebraic expressions. Second, its solution does not result from applying rutinary procedures. One possible approach that would lead to such an expression is realizing that any polygon of $n$ sides can be subdivided into $n-2$ triangles, each with a sum of internal angles of 180°. Since the polygon is regular, the internal angle results from dividing this total sum of angles by $n$:

$$Internal\ angle\ =\ \frac{180°(n-2)}{n}=180°-\frac{360°}{n},\ n\geq 3.$$

However, the task is stated mostly as a computational problem. Students should command the *sprite* to draw any given regular polygon. Its solution (Figure 1) requires generalizing the number of *sides* as a variable, applying a *repeat* loop and implementing a corresponding turning angle. Moreover, the thought experiment for such an expression is rather different than the mathematical approach. One imagines a *sprite* walking a full circle of a low resolution. As such, the total turn of 360° is divided into the number of sides. The one mathematical operation involved is therefore a division.


**Figure 1: Scratch script section for a generalized regular polygon**

Students tackled this problem through an open-ended exploration. A group of students use a repeat-10 loop turning 15 degrees, producing the arch of a low-resolution circle.

| | |
|---|---|
| Andy: | It has to be repeated 25 times. |
| Teacher: | Why, Sam? |
| Sam: | If we say [repeat] 10 once more, then it is almost made, then a small part is missing. And then, [we repeat] 5 [more] to make it. |
| Teacher: | You think that happy mouth could become a circle maybe, or what? |
| Hector: | Is it 24 times? |
| Teacher: | Why? |
| Hector: | Because then it divides up into 360 [degrees]. |

Hector's discovery makes use of a mathematical operation to solve a computational problem. They are not exploring or visualizing new mathematical ideas when using the digital tool. From an MDC perspective, students should take advantage of the tool's pragmatic and epistemic values (Artigue, 2002; Geraniou & Jankvist, 2019). For these students, the computational tool has mostly a pragmatic than epistemic value for the sake of mathematics learning. Programming served as a way of skipping learning about internal angles to solve a graphical problem.

The third part of the teaching sequence, that of drawing skylines in Scratch, exacerbates the disciplinary issue. The teacher's ambition was that students could pick their own city skyline to draw, making use of what they learned about coding, angles and polygons. They began by drawing with pen and paper, adding a sketch of how they would instruct Scratch to draw it. Figure 2 displays one such case, where a student engaged significantly in drawing the Statue of Liberty, and sketched unprecise instructions on its coding. These commands, at most, include directions such as "up" and "turn," without even specifying angles.


**Figure 2: A student's sketch of the Statue of Liberty**

For the student, this part of the task presents an amusing graphical challenge, to be eventually solved by computational means. Her mathematical knowledge is not evident, despite the experience with the polygons part of the task.

Overall, the empirical basis displays three domains: mathematics, computational thinking, and graphical design or technical drawing. In their recent study, Sand et al. (2022) devise this triad as having three nodes: mathematics, code and output, and its interconnections as acts of modeling, yet another essential aspect of computational thinking (Ejsing-Duun et al., 2021). The teaching unit has the potential to confront students to solve a geometrical abstraction problem (mathematics) by modeling it as the trajectory described by a sprite on a screen (output), commanded by a block-based program (code).

In the context of strictly mathematical competencies, Niss and Højgaard (2019, p. 15) draw a line stating that "the problem handling competency deals with intra-mathematical problems only". However, they acknowledge the interdisciplinary issue:

> It may happen that a problem has arisen from extra-mathematical needs or domains, i.e., by way of mathematical modeling. However, it is only problems in their mathematical instantiations that are covered by this competency. (pp. 15–16)

This is consistent with Kallia et al.'s (2021) view on CT in mathematics classrooms as a structured approach to solving mathematical problems, with (computational) modeling among the thinking processes to go about them.

Therefore, we argue that a path consistent with the literature is to use effective computational modeling of mathematical problems that may be anchored to an external output. In this sense, the polygon task has the potential of exploiting this aspect. The skyline task, particularly in the case of the student's version of the Statue of Liberty, fails to engage in handling a mathematical problem. This bypass is due, in part, to the freedom given to the students in choosing their own skylines of reference to code, which leads us to the next aspect.

## Ownership of the problem

The second issue regards ownership in the problem formulation. The question is, then: whose problem is it? This aspect may refer to the meaning of a problem, as opposed to any other procedural exercise. Niss and Højgaard (2011) referred to the meaning of a problem being relative to the person facing it, and we tackled it in the previous section, based on the procedural nature of the general expression for the angle. More importantly, we are concerned about the ability to pose—and not only solve—problems.

Our original formulation of the task was produced so that it avoided difficulties for the students (Elicer et al., 2022). In particular, we included a working script as point of departure and a research-based sequence of polygons in increasing order of difficulty: square, triangle, hexagon, pentagon. The teacher challenged this approach during the pre-intervention interview:

Teacher: It is easy to do it. But (…) I would start just with the pen and then tell the children: "how can you code a triangle?" So, instead of giving the code from the start, the first thing is to let the children make it themselves. Maybe they can do it; maybe they can't do it. But they need to understand what we are doing now. Instead of just having the code, [let's ask] "how can we do this?", "what is the problem?", "what do we need to know?", "what kind of code do we need to do to code a triangle?"

Since the task's first iterations, the teacher insisted on framing the problems so that students put their motivation and interests in the problem to solve, which led to the skyline part of the task. She made this point very clear during the pre-intervention interview:

Teacher: They can program it into something that gives meaning in their lives. If they like football, they can do it in football. If they like fairy tales, they can do it in fairy tales. I don't care. It is *their* digital production. It is *their* expression. And that is important.

The students' ability to pose problems is then exercised by allowing them to make choices in the graphical characteristics and sequence of polygons, as well as the skyline of reference. In the first part of the task, the choices do no harm the learning goal of finding a pattern in the angles. In the second part, the experience varied. Another student chose to code the Brandenburg Tower in Berlin. In Figure 3, one can see his sketch, clear steps to code it, and its successful implementation in Scratch. During the post-intervention interview, the teacher emphasized that his ownership of the problem played a role in his success:

Teacher: He set [out] to make his code and he could do it exactly as he has done on the paper [with the] Brandenburg Tower in Berlin. He has been there, and it was important for him to do it in a childish way (…) He has had a good experience and then he used that into the coding and the math.

In a way, the teacher doubled down on her approach to let students pose their own problems. However, as shown in Figure 2, these choices can lead to disregarding the possibility to solving them through the available mathematical and computational tools.

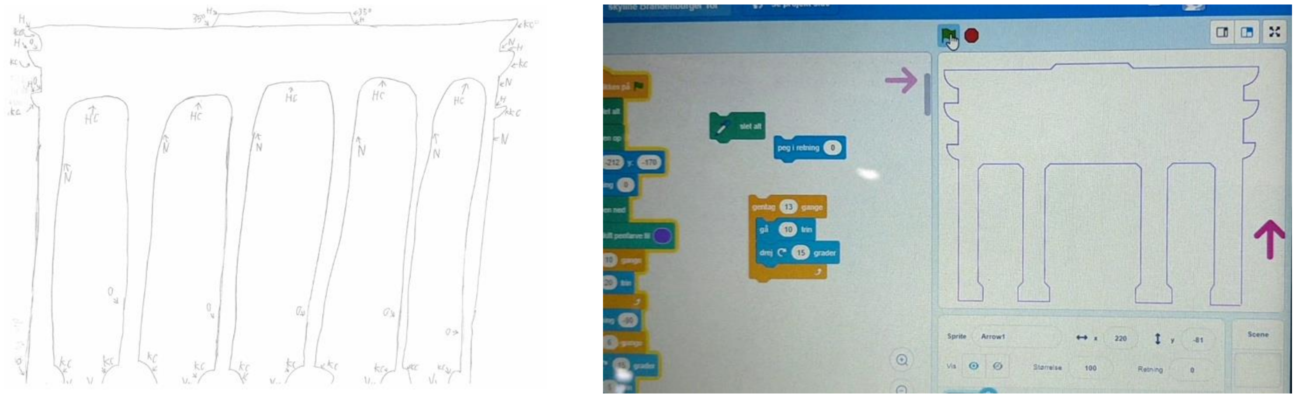


**Figure 3: A student's sketch (left) and coding (right) of the Brandenburg Tower**

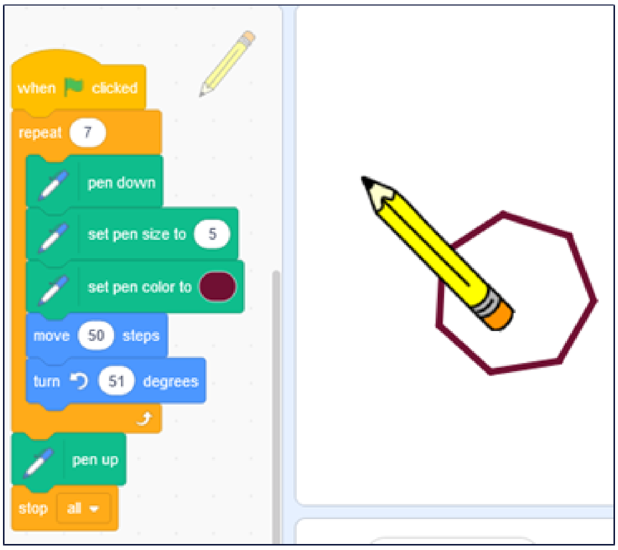
Within the *computational problem solving practices* branch of their taxonomy, Weintrop et al. (2016) begin with *preparing problems for computational solutions*, admitting that "while some problems naturally lend themselves to computational solutions, more often, problems must be reframed" (p. 138). In the skyline task, the teaching takeaway is that the degree of freedom given to students should consider the possibility to make abstractions and models (Ejsing-Duun et al., 2021) suitable for computational solutions, i.e., straight lines, polygonal figures and circle arcs. In that sense, we propose that Kallia et al.'s (2021, p. 21) long list of thinking processes involved in solving (sic) mathematical problems— "(not limited to) abstraction, decomposition, pattern recognition, algorithmic thinking, modeling, logical and analytical thinking, generalization and evaluation of solutions and strategies"—should be split. Abstraction, decomposition and modeling are more related to delineating a problem, while pattern recognition, debugging and evaluation are rather connected to its solution. The role of CT as a set of ways of thinking when posing mathematical problems should then be carefully distinguished and focused.

## Transferability

The third aspect is that CT is meant to provide solution strategies transferable to another person, machine and even discipline (Kallia et al., 2021). This feature is a consequence of CT's broad scope that proponents have highlighted, i.e., its potential to solve problems in a wide variety of contexts (Li et al., 2020; Weintrop et al., 2016; Wing, 2006). This aim of building general problem-solving strategies is common to mathematics education, and there is a risk of mathematics losing that territory, as Djorgovski (2005, p. 130) proposes:

> Another, mildly provocative idea, is that applied computer science is now playing the role which mathematics did from the 17th through the 20th centuries: providing an orderly, formal framework and exploratory apparatus for other sciences.

Back into the classroom experience, the issue of transferability is most illustrated by the use of two solution approaches to the polygon problem. After coding different polygons in Scratch, the teacher asked students to open GeoGebra and replicate the construction of different polygons. GeoGebra has features to draw regular polygons and measure their angles easily. However, the angles involved are not the same as the turning angles used in the Scratch solution, as one can see in Figure 4 regarding the heptagon. On the left side, the turning angle in Scratch represents the external angle; on the right side, GeoGebra is displaying the internal angles.



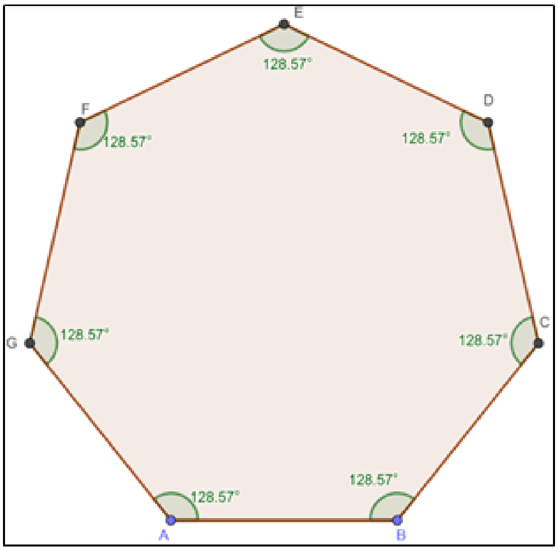




**Figure 4: Scratch (left) and GeoGebra (right) approaches to a heptagon**

These angles are, of course, supplementary, i.e., they add up to 180°. However, the issue becomes less equivalent when it comes to finding a pattern leading to a general expression for these angles. For this purpose, the teacher asked students to collect the angles and sums of angles on a shared Excel sheet (Figure 5). Students not only struggle more in finding such a pattern in GeoGebra, but they do also not find the point of this addition to the task. As described in the excerpt involving Danny and Hector, it makes sense for them to visualize the (external) angle of a polygon as a subdivision of circular trajectory that adds up to 360°. The meaning of an angle in a CT environment is the preferred provider of a solution strategy over Euclidean geometry. Transferring or translating these approaches is possible but seems unnecessary.

| Kant | Vinkel Scratch | Vinkelsum | Vinkel GeoGebra | Vinkelsum |
|---|---|---|---|---|
| 3-kant | 120 | 360 | 60 | 180 |
| 4-kant | 90 | 360 | 90 | 360 |
| 5-kant | 72 | 360 | 108 | 540 |
| 6-kant | 60 | 360 | 120 | 720 |
| 7-kant | 51 | 360 | 128,57 | 899 |
| 8-kant | 45 | 360 | 135 | 1080 |
| 9-kant | 40 | 360 | 140 | 1260 |
| 10-kant | 36 | 360 | 144 | 1440 |

**Figure 5: Excel screenshot with comparison between Scratch and GeoGebra**

In the early years of CT as a pedagogical concept, Papert (1980) refrained from framing it as a new approach to learn the already established mathematics syllabus. "Euclid's is a logical style. Descartes' is an algebraic style. Turtle geometry is a computational style of geometry" (p. 55). CT was framed as a new and versatile means for students to produce alternative mathematical ideas.

The lesson is that the transfer between approaches, be them computer languages, digital tools, contexts and disciplines, must consider that each domain carries its own codings and registers. This concern is consistent with other empirical studies such as Cui et al.'s (2021), wherein primary school students' main challenges stem from the differences between CT and mathematical thinking. These must be accounted for when aiming for developing modular solutions with an "ability to be easily reused, repurposed, and debugged" (Weintrop et al., 2016, p. 139). Otherwise, it would contradict the view of a CT as a robust strategy that handles problems "so that they do not create new problems" (Ejsing-Duun et al., 2021, p. 426).

## Conclusion

Mathematical PS is commonly framed as a main feature of mathematical activity (Schoenfeld, 1985), becoming both a means to learn mathematical notions (Cai, 2010) and a goal in itself such as the problem handling competency (Niss & Højgaard, 2011, 2019). The widespread inclusion of digital technologies is consistent with their subordination to aid mathematical problem solving (Geraniou & Jankvist, 2019). However, we have argued that CT, another construct in a digitalized society to enter the mathematics education agenda, may play a critical role in dealing with mathematical problem handling.

Based on definitions from the literature and anchored in an empirical basis, we propose that a CT-driven mathematical problem handling competency should entail, at least the following abilities:

- *To solve computational and mathematical problems* with, respectively, mathematical and computational solution strategies through effective modeling. Problems may arise from extra-mathematical contexts (output), but this competency is displayed when it leads to a non-rutinary mathematical challenge (Niss & Højgaard, 2011).
- *To pose own real-life inquiries as computational and mathematical problems*, with an awareness of which elements can be selected (abstraction) and approached by computational solution strategies (modeling). This aspect requires a focus on CT processes (Kallia et al., 2021) that aid the formulation of problems into computational terms.

- *To judge which elements of the solution strategies can be transferable*, and seizing ambivalences as epistemic obstacles for the learning of mathematics and CT. This element is connected to the problem of CT emerging from a different domain—computer science (Papert, 1980)—, the necessity of building a tolerance for ambiguity (Pérez, 2018), and the opportunities to reuse and repurpose computational solution strategies that do not create new problems (Weintrop et al., 2016; Ejsing-Duun et al., 2021)

Our case study and our conclusions engage with what *may* be possible from a competence standpoint. As Niss and Højgaard (2019, p. 21) acknowledge, "full mastery of a competency is a 'point to infinity'". We believe these three points can provide guidance toward designing learning environments and assessment activities that showcase what students *can, in fact,* do.

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# Fostering Computational Thinking: The role of understanding mathematical concepts in the context of debugging computer codes

Jacques Kamba[1] and Viktor Freiman[1]

*In New Brunswick, several schools offer opportunities for students to learn engineering design associated with coding and physical computing. However, there appears to be a lack of effective pedagogical interventions to support the development of skills associated with debugging among these students. This article shares the first results of a doctoral study, still in progress, aiming to present the state of the problem related to students' debugging practices during the realization of various creative projects that involve coding with the use various physical computing devices. It specifically reflects on the role of understanding mathematical concepts while debugging code errors when creating engineering artifacts in a school-based STEM lab.*

*Keywords: Debugging, coding, physical computing devices (PCD), mathematical concepts, Computational Thinking (CT).*

## Context of the study

In the past decades, an increasing attention of educators has been directed to attracting more K-12 students to disciplines of science, technology, engineering and mathematics (STEM). Among a variety of learning experiences serving this purpose, involving students in interdisciplinary projects integrating designing artefacts, computer coding, and physical computing devices (PCD) was associated to the development of Computational Thinking (CT) as a novel approach to solving problems (Wing, 2006) in an authentic/real-world context (Jona et al., 2014). One such project was a part of a case study of school makerspaces and other types of making activities conducted by the CompeTI.CA (Compétences en TIC en Atlantique/ICT Competencies in the Atlantic Canada) research team in several schools in New Brunswick (NB), Canada (Freiman, 2020). This article reports data from this study featuring a group of Grade 4 students from one primary (K-5) school. The group was invited to work on a science project investigating animals. Students had to search for information about an animal and make a presentation to sensibilize their peers, parents, and members of their community to the issues of protection and conservation of species, some of which being endangered. Part of the project was a CT assignment for which the students built a physical model of the animal with some parts of its body making rotating motion by using a *micro:bit-operated servo motor* with an integrated computer code.

During the design of their prototypes, students faced different issues related to building a physical model, computer coding, transferring it to a *micro:bit,* and then incorporating a programmed PCD into their model to ensure that selected parts make an appropriate movement. In this article, we are interested in how young learners attempted to debug their codes, while resolving issues relating to PCD and its installation on their physical models. We will then attempt to relate our findings to the

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debugging as part of the development of CT while looking into possible connections to mathematical concepts and applications, as well as higher-order problem-solving skills and processes. Debugging processes and the undertaking of related decision-making are not yet sufficiently clarified in the literature on STEM education (LeBlanc et al., 2022). The following section describes the context of the study. Then, we briefly introduce key theoretical concepts underlying the study followed by the task description, methods, and partial results related to debugging codes.

In New Brunswick (NB), Canada, the last decade has been marked by a noticeable enthusiasm among some teachers to engage students in an integrated STEM learning in school makerspaces while introducing them to a variety of cutting-edge technologies (Freiman, 2020). In order to help teachers in setting up making activities to stimulate students' interest in STEM, several not-for-profit or charity associations, such as Brilliant Labs (www.brilliantlabs.ca) or Place aux compétences (https://pacnb.org/fr/) have been providing teachers with different resources (materials, including latest technologies, and on-site assistance). There are a variety of forms and settings in which making activities are conducted. Some teachers let students freely choose a project according to their interests. Others would integrate making into a more formal, subject-related teaching, as was the case in our study where the teacher decided to incorporate computer coding and physical computing technology-related, design-based projects suggested by the provincial primary science curriculum.

Our research team, in collaboration with Brilliant Labs, has been exploring students' learning in makerspaces from different perspectives. These include creativity development (Freiman & Robichaud, 2020; Léger, 2022), different approaches to problem-solving such as tinkering (Furlong & Léger, 2022), and exploration of various technologies such as 3D-printing (Freiman & Kamba, 2020). A recent paper by LeBlanc et al. (2022) highlights a dialectic interaction between hands-on and mind-on learning opportunities that making activities might provide for applying mathematical concepts and operations, while targeting higher-order mathematical processes such as reasoning. In this context, we identify a lack of research on the role mathematical connections might play during processes of debugging, as part of CT practices. Such processes are to be developed in a variety of design activities going beyond programming (Brennan & Resnick, 2012), which we extend toward a process of designing and testing a physical computing system viewed as a whole. In the next section, we discuss this process and possible mathematical connections in more details.

## Conceptual framework

In relation to the problem of this study, the conceptual framework clarifies different stages of an engineering design process into which a construction of a physical computing system is embedded, as well as their connections to CT, debugging and mathematics.

## Engineering design process

When working on their prototypes, students become engaged in a design thinking cycle as part of "a human-centered approach that relies on innovation, collaboration and creativity to solve a multitude of social or environmental issues" (Pruneau et al., 2021 with reference to IDEO.org, 2012). Each cycle usually consists of several stages, namely (1) observation-inspiration, (2) definition-synthesis, (3) ideation, (4) prototyping, (5) testing and (6) communication. Adapted to the context of our study, stages 1-2-3 are combined in a Step 1 (Figure 1) where students having already built a physical model of their animal, begin their process of design of a physical computing system by choosing a part of the animal's body to be dynamized (observation-inspiration) and imagining how this part would move and how it could be automatized (definition-synthesis-ideation). As the second step (stage 4), a prototype of a physical computing system is built including coding, transferring to a PCD, and integrating it into a prototype. It is followed by the stage of testing (Step 3), and public presentation of the project (Step 4-stage 6). When going through the process of designing and testing their prototypes, students can face a number of code errors and issues related to PCD that would require debugging.


**Figure 1: A 4-step design process**

## Integration of PCD in the design process: Interconnection between physical model, physical computing devices, and computer coding

Physical computing (also called computational making) frames students' projects as an approach to designing real-world (physical) interactive objects by implementing knowledge from computer science, electrical engineering, and related disciplines. This is made possible through the use of hard (LEDs, sensors, servos) and software components (source code) (Juškevičienė et al., 2021). Juškevičienė et al. also emphasize the role of physical computing in the development of higher mental functions as physical computing affords humans ways to co-construct their environments thus mediating coordination of social and intellectual functions to support learning. In our study, a physical computing system built by students can be represented by the following schema adapted from Jang and Kim's (2016) framework of interworking between physical models, digital models, and physical computing (Figure 2). In this system, a servo motor controlled by a microprocessor micro:bit running a program (code) produces a desired rotating motion when integrated into a physical model. A virtual component of the system incorporates a code that students created using a MakeCode platform with an option to use a JavaScript or a block-based language, more suitable for young learners, to describe the desired algorithm of rotational movement. The codes were then transferred to a PCD (micro:bit) to which a servo motor was then connected. A data flow was nonlinear providing different types of feedback about its functioning. This is where several types of issues can be detected such as problems with the code, with PCD, or with the physical model. Dealing with these issues engages students into a debugging practice as part of a CT development.

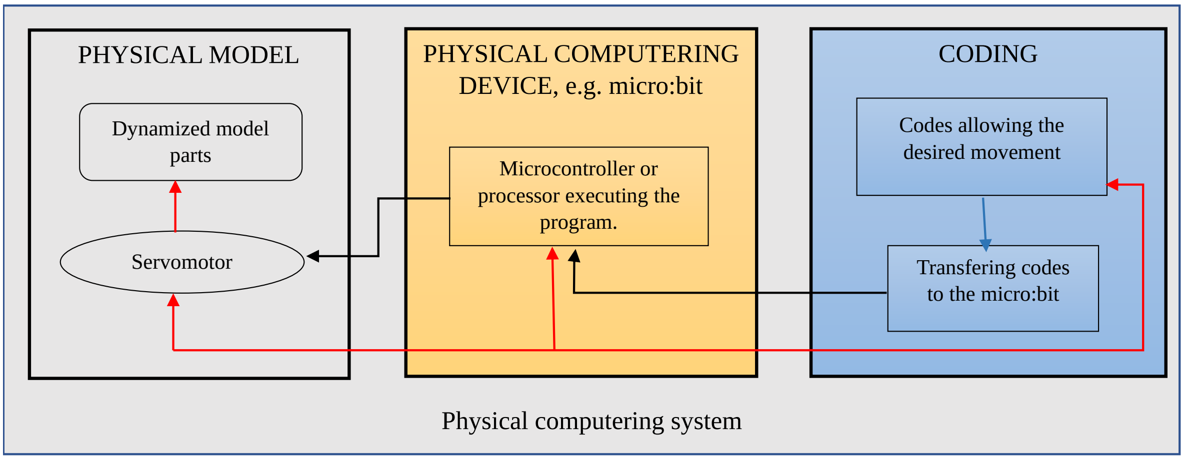


**Figure 2: Interworking between physical (moving part of an animal) and digital models (code) by means of PCD (*micro:bit*) (adapted from Jang & Kim, 2016)**

## CT and debugging practice

Computational Thinking (CT) is still an evolving concept that is articulated by a diversity of definitions and aspects (Brennan & Resnick, 2012). In the 1970-80s, Papert provided the LOGO-based programming. This was used as a tool not only to support mathematically rich activities but also to engage "novice and expert, young and old" in the construction of educationally powerful computational environments that will provide alternatives to traditional classrooms and traditional instruction" (p. 182). The term CT was reinvigorated by Wing in 2006, as a "fundamental skill for everyone," along with reading, writing, and arithmetic, to be added to "every child's analytical ability," and as a way of "solving problems, designing systems, and understanding human behavior, by drawing on the concepts fundamental to computer science" (Wing, 2006, p. 33). Wing (2008) identified abstraction, logical thinking, algorithmic thinking, innovation, and creativity as central elements to the constitution of CT. For Lee et al. (2020), CT "is the thought process involved in formulating problems such that their solutions can be expressed as computational steps or algorithms to be carried out by a computer" (p. 1).

In their work, Brennan and Resnick (2012) have proposed a CT framework consisting of three dimensions: computational concepts–the common concepts used in programming such as loops, events, conditionals, repetition; computational practices, the process of programming such as testing and debugging, reusing and remixing, abstracting and modularizing, and computational perspectives, students' understandings of themselves and the technological world. Within a larger scope, Shute et al. (2017) conducted a research review on definitions of CT in different disciplines and from different perspectives. They came up with their own transdisciplinary view of CT as "the conceptual foundation required to solve problems effectively and efficiently (i.e., algorithmically, with or without the assistance of computers) with solutions that are reusable in different contexts" (p. 151).

In connection to engaging learners with physical computing, Juškevičienė et al. (2021) have emphasized that "artifact development, idea modeling and experimenting with tasks (mini projects)" help students to focus on "practical experiences of CT to think computationally" (p. 178). Indeed, anchored in Papert's constructionism, several authors highlight benefits of "physicality as an essential link between children's embodied experiences in the world and the new universe of computer code" (Horn & Bers, 2019, p. 663). Moreover, the authors argue that "learning experiences that make use of physical properties of materials, movement through space, and relationships between objects and people might more successfully reference sensorimotor schema that form the foundation for much of abstract thought" (p. 668). This type of experience creates, in turn, new perspectives on debugging.

Although CT definitions and structures vary and are still subject to debate among authors (Denning, 2017; English, 2017; Shute et al., 2017), most of them include some aspects of debugging as an essential practice for the development of CT. As is it in the case of CT, debugging is also defined in many ways and from different perspectives. From a learning to code perspective, Heikkilä and Mannila (2018) define debugging as "a problem-solving activity, which involves exploration, observation, communication and reflection" (p. 2). In this definition, debugging is presented as a process of correcting code errors and solving problems with digital tools, supported by reflection, exchanges with others and exploration. Vasconcelos et al. (2020) point out that debugging is also known as troubleshooting. They defined it "as the process of identifying error(s) in a program and using problem-solving strategies to fix it" (p. 64). Debugging is different from other approaches such as trial and error, as it is a systematic and thoughtful process in which students identify and fix a code error by applying a corrective approach and testing hypotheses until the error is fixed (Vasconcelos et al., 2020).

It is also a process of figuring out why a program is not working or not behaving as expected. While a person debugs a problem, they can identify the issue, test the program to isolate the source of the error, and reproduce the problem so that potential solutions can be tested reliably (Weintrop et al., 2016). McCauley et al. (2008) argue that debugging is a process that requires some steps and approaches to understand the error and correct it. Among the different approaches used by young children to debug a code error, Bennie (2020) and Deliema et al. (2019) have proposed the approach of requesting assistance to get some help, advice or clues from a teacher, a resource person, or a peer to debug the program. On this approach, Brennan and Resnick (2012) point out that interactions with the teacher or with peers allow young students to obtain information needed to debug their codes.

In a context where learning to code is integrated into a design of physical computing systems, Hennessy et al. (2023) defined debugging as "a situated inquiry where students develop an iterative process of understanding as they tinker with the software and hardware, leading to developing perspectives on the system as a whole" (p. 6). In the context of our study, this approach can be used not only to help students to debug their codes, but also to solve problems related to the PCD and physical models (prototypes) with integrated computing devices (in our case, *micro:bit* connected to servo motor placed in a moving part of the animal's body).

## Mathematical connections related to CT and debugging

By transforming a classroom into a project-based space for making, the design thinking process allows young students not only to review different mathematical concepts, such as measurement, scale and area, but also learn how they can be applied to solving real-world problems (Kim et al., 2013).

According to the literature, there seems to be a strong connection between CT and mathematical thinking. Shute et al. (2017) explain that "mathematical thinking consists of three parts: beliefs about mathematics, problem-solving processes, and justification for solutions" (p. 4). In its turn, computation, along with CT, refers to computational models as mathematical abstractions where the thought process is represented as computational steps and algorithms. Hence, algorithms are one of key elements explicitly relating two disciplines, mathematics and computer science (Aho, 2012), which is viewed as succession of actions arranged in a well-defined order to perform a task or solve a problem (Nguyen, 2005). Already in 1960s-80s, an emphasis of computer scientists was put on a concept of algorithm as a "whole range of concepts dealing with well-defined processes, including the structure of data that is being acted upon as well as the structure of the sequence of operations being performed…by machines" (Knuth, 1985, p. 170).

Knuth also refers to the Soviet-bound conception of cybernetic (Kibernetika) and its relation the control of a (computational) process or applied mathematics (Prikladnaia Matematika) thus "emphasizing the utility of the subject and its ties to mathematics in general" (Knuth, 1985, p. 170). In our study, algorithms were not explicitly taught to the students. When building their physical computing systems based on interactions between physical models, PCD, and computer program (codes), children have rather intuitively dealt with algorithms through a variety of representations of desired movements as objects (parts of their physical models), computing devices (servo motors and *micro:bit*s) to control and execute the movements, and a virtual code (MakeCode blocks of codes), as we will describe in the next section.

Introducing young students into physical and virtual computing emphasizes yet another possible connection of CT to mathematics, also documented in the literature is related to spatial explorations and more specifically, geometry. For instance, in his article entitled "An exploration in the space of mathematics education," Papert (1996) explored CT in relation to children's learning of mathematics, in particular geometry. "The goal is to use Computational Thinking to forge ideas that are at least as "explicative" as the Euclid-like constructions […] but more accessible and more powerful" (p. 13). Already in the 1980s, Battista and Clements (1988) argued that elementary geometry should be the "study of objects, motions and relationships" whereas the primary goal is "the development of students' intuition and knowledge about their spatial environment" (p. 11). In this respect, according to the authors, introducing computers and programming, at that time using Papert's Logo, was considered as a meaningful tool to facilitate translation of children's intuition into a more precise language of commands where the focus would be put on geometric concepts and their properties (like conceptualizing a rectangle as a shape with angles of 90°) (Battista & Clements, 1988 with reference to Papert, 1980). This vision is anchored in Piagetian insight about pertinence of physical actions.

Indeed, according to Piaget and Inhelder (1967, cited in Battista and Clements, 1988), the child "can only 'abstract'...the idea of a straight line from the action of following by hand or eye without changing direction, and the idea of an angle from two intersecting movements" (p. 14). In the 21st century classroom, this perspective has only gained in importance in context of CT development using modern programming tools (e.g., Scratch) and devices (e.g., EV3 robots) where the emergence of spatial reasoning is stimulated by simultaneous development of enactive–iconic–symbolic representations (Francis et al., 2016 with reference to Bruner et al., 1966). In a similar way, Ramey et al. (2018) emphasize specific interactions between people, tools, and representations through which spatial thinking is enacted and developed to trace specific spatial representations as they traverse across representational media, to understand how spatial understandings are distributed to or co-constructed by learners and their socio-material context, such as an engineering design.

Finally, recent studies of CT development as a context to enriching mathematics teaching and learning, in connection to Chevaillard's praxis-logos didactical interactions when transposing knowledge have allowed to identify possible ways to reinforce early algebraic thinking while working with computer codes, for example, in Scratch (Kilhamn et al., 2022). The authors have specifically analyzed how the coding task that involves variables (e.g., changing values of angles in 'turn' blocks) could be didactically transposed to make 'algebra logos' explicit (Chevaillard, 2006; Kilhamn et al., 2022).

From the mathematical perspective, several CT thinking skills, such as sequencing (algorithms), modularity, and debugging could be considered 'foundational' to mathematical understanding, already at a very young age (4-5 years old) (Lavigne et al., 2020). The authors argue that when children are engaged in the debugging (which means identify the problem, break it down into smaller parts, and test solutions), they can use mathematical knowledge (such as using a pattern, counting, or comparing) to correct the error (Lavigne et al., 2020). In their turn, according to Kilhamn et al. (2021), teachers believe the debugging practice can encourage students to "see failure as a natural part of a problem-solving process" which makes programming activities beneficial for mathematics learning (p. 174).

With this theoretical perspective in mind, we are particularly interested in students' conceptualization of rotational movement of a servo motor as a focus of designing, testing, and debugging their computerized system. In this respect, we reviewed a few studies that may potentially reveal connections between mathematics learning and debugging. For instance, Shumway et al. (2021, p. 21) have studied mathematical concepts relevant to the process of debugging. They have shown that the mathematical concepts of counting, spatial reasoning, units of measurement and operations knowledge have emerged as helping or hindering forces when students debugged programs. Also, Rich et al. (2019) showed how engaging students in a debugging cycle as they observe, hypothesize, modify, and test, can create a fruitful context to mutually support mathematics learning and CT development in an integrated way. In a similar way, Kilhamn et al. (2021) have studied the potential of computer programming, when integrated into mathematics curriculum, to support exploring problems and mathematical concepts.

## Guidelines of the methodological framework

Our study is situated in a qualitative and exploratory epistemological posture, focused more specifically on an inductive interpretative approach. This paper describes a case which is a part of a larger doctoral project conducted by the first author based on grounded theory (GT), as described by Luckerhoff and Guillemette (2012). A method of case study, as advocated by Creswell and Poth (2018) aimed at better understanding the debugging process followed by NB students in the context of digital creation (incorporating coding and PCD) in the classroom. Two cases are featured: (1) Grade 4 students working on designing an animated animal model with *micro:bit* and servo motor integrated in a part of the model making rotational movements; and (2) Grade 4 students creating an interactive story book incorporating sounds programmed using Scratch and *Makey Makey*. As researchers, we were interested in their debugging process demonstrated during the whole design process (Figures 1 and 2). In this paper, we present some partial data of one team from Case 1 (animal model) to learn how students are solving a coding problem to make a part of the animal move. The analysis process applied in the context of this article is the same as planned for all the data from the doctoral study, i.e., an analytical process of codification and categorization (Paillé, 1994) taken from video recordings, interviews with students and researcher's notes taken during field observations. The data collection has been reviewed and approved by the university research ethics board. All participating students have submitted a form signed by their parents consented to their child's participation in the study.

## Partial results of the study: Case 1, team 1

During the design process, we learned about the work done by the students from Grade 4 class. The students were grouped into a small team of three people. They designed and created a model to solve an ecological problem about an endangered animal, named spider monkey. One part of this model (the tail) was supposed to be energized to exercise a rotational 180-degree movement using a *micro:bit* and a servo motor. The symbols S1, S2, S3 identify each student on the team, the symbol RS represents the researcher, the symbol TC represents the teacher, and the symbol RP represents the resource person (a mentor from a partnership organization helping teachers and students to realize their models, especially in the coding part).

First, the students seemed to be well aware of the design process and could verbalize their ideas. One of the students (S1) showed the part of the model that was supposed to make movements: "But it's his tail that will move (the tail of the model)." Another student gave more details on the tools to be used to make the tail rotate: "Hey, the *micro:bit*, the helicopter technology (the servomotor propellers, JK and VF) that will do that on their own...." They have established the location and taken measures to achieve the attachment of the tail to the model as shown in Figure 3 below.

**Figure 3: Animal model, the location of the tail**

The below conversation with the students helps to learn about their representation of the whole design process and the role of technology in it.

RS: How is it going to move? What will move? Is it the mouth, the head or what is it?

S2: *It's the tail. We haven't glued it yet to put the technology in it. Then, there we will glue it* (She shows the place where the tail would be glued.) […].

S1: *As we told you, we are going to make technology move it*.

RS: Are you going to drill down here to put in the tail?

S1: *We're just going to stick it, we're going to put the technology (micro:bit ) in it.*

During the conversation, one of the students was making gestures with one hand (red arrow) to show their intuitive algorithm ideas of movement of rotation as shown in the table below (Table 1).

Indeed, students' intuitive understanding of angles, movement, and direction was not accompanied using the relevant mathematical terms but an understanding of what needs to happen was apparent. Afterwards, using a MakeCode coding platform, students attempted to create the sequences of the computer code (blocks) operating the movement of their animal's tail to be executed by the servo motor controlled by the *micro:bit*. When students were trying to create a code representing this movement, their first version of the program contained three blocks of code: an activation block (once you press an A button on the *micro:bit*, the program will be activated), a block for setting up an initial position of the servo horn to 180° (servo write pin P0 to 180), and the pause block (pause 100 ms), which must be introduced between each rotational movement of the servo motor, as shown in Figure 4. Interestingly, when testing their program, students realized that the program movement did not correspond to their intuitive idea of the algorithm, which could be due to a code error. One of the students shared with us their observation:

S1: *Usually when we press A, it works; there, it didn't work anymore.*

Seeing that the servo was not moving at all, students had to identify several incorrect/missing parts of their code: (1) the first pin block was supposed to set up a value of the angle to 0 degrees; then (2) have a second pin block set up to 180 degrees thus controlling the rotational movement of the servomotor of 180 degrees (desired rotation); (2) add a loop block to allow the program to repeat the rotation movement a desired number of times; (3) increase the duration of the pause to 1 second (1000) milliseconds (ms), (the value of 100 ms is not enough to ensure the movement of propeller producing a desired angle); (4) insert another block of code for the pause for the return movement, then to make sure the movement is repeated several times (loop). So, they started the debugging process to fix their program.

**Table 1: Representation of intuitive algorithm ideas**

| Data: The gesture of the hand to represent the intuitive algorithm idea | Analysis |
|---|---|
|  | In these three images, student S1 (the hand with a red arrow) tries to represent the movement algorithm they would like to implement in order to allow the *micro:bit* and servo motor to control and to produce a rotational movement of the tail.<br><br>According to our observations of this hand gesture in the video recording data, the student anticipated that the animal's tail will be rotated clockwise in a semicircle, going roughly from 0 degree to 180 degrees, as shown in the images on the left. However, the student did not verbalize the precise measurement of the angle of rotation of this movement. |

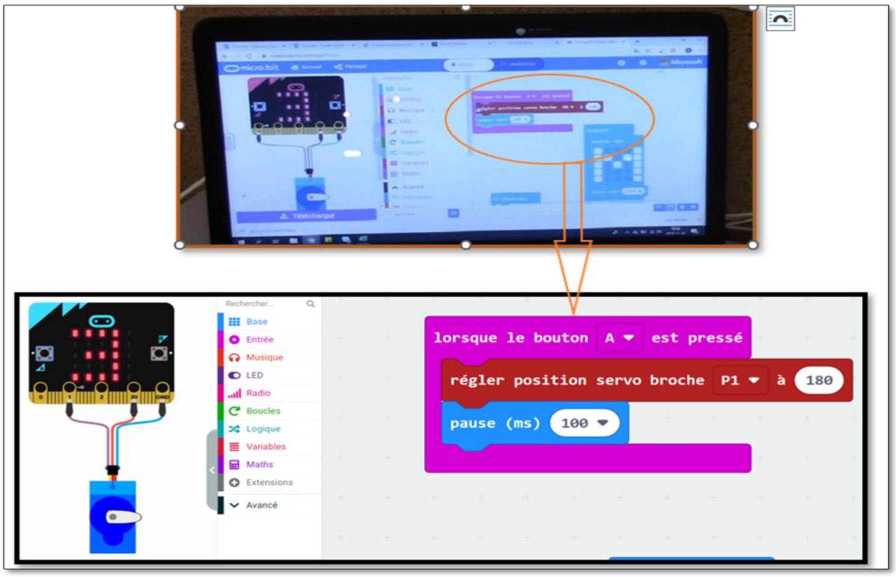


**Figure 4: Students' efforts to code tail movement using Makecode platform**

When asked how they proceeded to find the error, the students explained that they tried to use little tricks, like inserting some (other) 'small things' (blocks of codes) while deleting the others:

RS: How did you go about finding the code error in your program?

S5: *… well, we tried other things that we had, like little things (code blocks), we could try other things like, we could remove them…… (If it does not work).*

The approach of 'trial-and-error' to review, modify, and delete certain blocks of code as well as the 'do-it-yourself' approach did not allow the students of this team to find the desired solution to the code errors. A student S1 makes this confession:

S1: *Because we tried, then it didn't work, then we tried, we tried, we tried.*

This was a moment where the students decided to turn to the adults in the room, a classroom teacher and a technology expert present in the digital fabrication lab, hoping they would find a solution clue to the code errors.

Firstly, the teacher guided the students to reflect, to identify by themselves the errors in code and to find a way of correcting them. Presented here is the conversation between the teacher and the students about correcting coding errors.

CT: You have to add others (the two pin code blocks) like this, one (a code block), which will be set at 0 degrees and the other at 180 degrees.

S1: (she adds a new pin code block).

S1: *Well, another one (the code block) like this?*

CT: No…Look, that's not a (correct) pattern. When X (the name of a student) puts this in the right place, what is missing for a correct pattern (desired movement)?

S1, S2 and S3 (saying simultaneously, after some thinking): *A pause.*

CT: Here you must find one block (of code assigning the value) to 0 degrees and one to 180 degrees, and press button A, to see if it will work.

S1: (a student has activated the *micro:bit* simulator button, however the program still was not working properly).

After having guided the students to find solutions to coding errors that occurred in their program, the teacher asked the students to think about the need to call on the resource person for possible solutions to the remaining code errors. The next exchange took place when the RP (a technology expert) was called for further help. One of the students (S2) explained the problem to the resource person:

S2: *This* (the propeller) *goes to that place (0 degrees), but it doesn't go back* (to the initial position).

The RP has invited students to a whiteboard, trying to explain them the way of thinking and using a variety of mathematical concepts relating to distance (of the movement) and its duration; the goal was making students understand why 10 ms is not enough to make a distance from the initial position to the final position (Figure 5):

RP: You know, uhh…I don't know how I'm going to explain this to you. Let's say you have a distance of 0 up to…, let's take up to 10 km, okay. It will take a while to get like this (time to leave from 0 to 10 km). Let's say it will take a second…to go from 0 to 10 km. But you see here (he points to the program). You took 1 hundredth of a second (which was, in fact, one tenth, J.K & VF), how many milliseconds does it take to make a second? Do you remember? In a second, so…1 second equals 1000 milliseconds, okay. And there, you said you covered the distance from 0 to 10 km in 100 milliseconds. It's not giving it (propeller) enough time for it to be able to get to its full distance, because you're not giving it enough time to get there. You need to give him more time to get to his distance. How long is it?

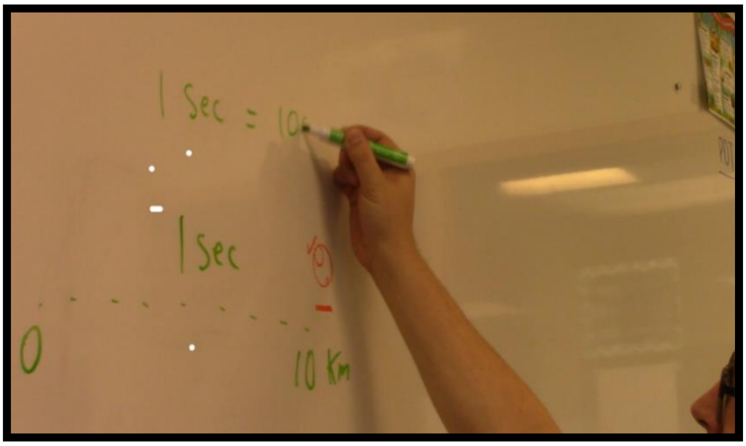


**Figure 5: Explaining to students the error code by evoking the concepts of time and distance**

So, knowing that 10 ms gives 1/10 of a second, the students had to figure out the time value to be inserted in the code equivalent to 1 sec.

S1: *One second* (the time to put in the pause code block)

PR: We can try to put one second (the PR encouraged students to try inserting this value into the 'pause' block to see if the program would work.)

S2: (She has changed the value of the time parameter in the 'pause' block to correct the code error (Figure 6) and then they tried the program, which finally worked correctly).


**Figure 6: Changing time variable in pause block**

## Preliminary discussion and conclusion

Our preliminary analysis of data is part of a doctoral study of the CT debugging practice during an engineering design process within a science project, which was realized by a group of Grade 4 students at one elementary (K-5) schools in New Brunswick, Canada. Specifically, our paper dealt with part of a debugging process where young (about 9 years old) children were introduced, for the first time, into a computer coding to program an animal part performing a rotational movement of an integrated servo motor controlled by a *micro:bit*. In particular, we looked to exemplify the role of mathematics in this process. Indeed, we can identify several episodes of students' work where mathematical connections were explicitly or implicitly present. For example, when students composed their code, they had to introduce different numeric values of variable identifying the initial position of a servo motor (at 180) and the intermediate position (at 0) while making sure the servo motor completes a back-and-forth 180 degrees rotation. In addition, they had to introduce a pause long enough to allow adequate time (1 second) to complete the movement. Yet, several problems have occurred requiring students' efforts to debug their code while paying attention to mathematical meaning of several components of their program.

In particular, when introducing the first pin writing block setting the initial position of servo motor, students have missed introducing the second one (intermediate position). This is why, during the testing, their virtual servo motor remained immobile once the movement button (A) was activated. Yet, from the mathematical point of view, this way of introducing a rotation (not yet seen in the mathematics curriculum by Grade 4) would require some sophisticated algebraic reasoning as was shown by Kilhamn (2022) but not yet made explicit to the young learners. In addition, besides a complex logic of the program requiring a pause block controlling the time for rotation, students seemed to be struggling in making sense of the parameter controlling the duration. Indeed, students of this age are not yet familiar with measuring duration in milliseconds. Furthermore, when talking about 'trying' to fix the program several times, children seemed to be unaware of a possibility to play with different parameters by changing the values of the variables involved. In fact, their first version was limited to the use of default values (180 for the initial position and 10 ms for the duration of the pause). On top of it, students could not realize that 1 sec is equivalent to 1000 ms. Only the intervention of the classroom teachers and a resource person helped them to complete the debugging process for this stage (coding and testing on virtual simulator).

We shared these very initial observations in our MACAS presentation. We leave the reader with the question about the role of mathematics in the debugging process, which requires further study. We can even be more provocative in terms of enriching discussion about connections of CT and mathematical thinking asking if mathematical debugging should be recognized as an essential part of the debugging practice and eventual didactical apparatus to support students' learning and in turn, as Kilhamm (2022) suggest, to make algebra-logos didactically explicit.

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# Breaking down classroom walls to STEMulate collaboration in science and mathematics education

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*There is a critical need for STEM education in schools. Yet teachers seem ill-equipped to do so, which suggests that part of the problem lies in pre-service education. We are therefore interested in critical issues in STEM education, which in turn leads us to ask curricular questions about how to integrate math and science in post-secondary education through a STEM perspective. During the Fall 2021 semester, the two professors teaching the mathematics education course and the science education course collaborated to engage students in a collaborative STEM experiential learning project. Interviews were conducted with three students after the end of the semester. The results suggest a strong potential for such collaborations to better prepare students for their future careers.*

*Keywords: STEM, mathematics education, science education, pre-service teacher education.*

## STEM: Why, how and where?

### Why?

In recent years, we have witnessed (or participated in) a movement aimed at developing students who are able to mobilize a range of knowledge, skills, and attitudes to better understand, analyze, and attempt to solve various problems around them, as well as actively participate in society by initiating or participating in innovative projects. Students need to make connections between science, technology, engineering, and mathematics (STEM) to find solutions to these problems (Bakırcı & Karışan, 2018; Bergsten & Frejd, 2019; Maass et al., 2019). There is no shortage of post-secondary students in STEM related fields. In 2020, the number of students enrolled in a science program had increased by 45% since 2010 and engineering program admissions were up 48% (Usher, 2022). Students in these programs develop a specialised set of skills, such as problem solving and critical analysis. However, as the labor market becomes more complex with automation and job transformations, higher STEM education is now in greater demand than ever before (Frenette & Frank, 2020). Students who emerge from the education system are expected to have learned integrative STEM. In many cases, this has led to the need to improve STEM education in schools. Yet, some people would agree that our current educational system does not really support STEM education (Gravemeijer et al., 2017). We suggest that the problem lies in the pre-service training of teachers, who feel ill-equipped to create learning-rich environments and support students in their STEM learning. While STEM education in schools is well documented, the situation is different for STEM education in the university setting. Although there are some university courses that address STEM, few initiatives leverage STEM education collaboratively across different courses. What type

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of pre-service training adequately educates future teachers in STEM education? What skills and abilities must they develop to be able to accompany students in their STEM learning?

## How?

Having true STEM education is not as simple as combining Science, Technology, Engineering, and Mathematics. Hobbs et al. (2018) describe five possible scenarios of STEM teaching. (1) Each of the disciplines is taught separately. For example, a student will have a course in biology, chemistry, physics, calculus, engineering, and programming all taught in parallel without any overlap. The idea is to give the basics and the student becomes responsible for the integration of the disciplines. Students are taught concepts independently such as the conservation of energy in physics, the energy levels of electrons in chemistry, and the ecosystems in biology and must determine how each concept relates to the others. (2) All four disciplines are taught but emphasis is placed on one or two. Often the two disciplines that stand out are the sciences and mathematics. (3) One discipline is integrated while the others are taught separately. For instance, mathematics is often used in physics and chemistry, and it can be easy to integrate mathematics into these scientific disciplines. Specifically, physics uses scientific reasoning as the basis for concepts, but uses mathematics to quantify them. Every law, whether it is the laws of motion, the laws of thermodynamics, or the laws of electrical circuits, can be represented and quantified by mathematical equations. Physics and mathematics are almost integrated by nature. The same is true of chemistry, which often uses exponents, orders of operations, algebra, unit conversion, and scientific notations to represent the concepts discussed in chemistry courses. Mathematics is integrated with physics and chemistry by its very nature. (4) The STEM curriculum is divided into the separate disciplines. A teacher may present a problem and the students will need to look at it with different lens. The solution will vary depending on the discipline the student is using as a lens. (5) Total integration of the four disciplines. It becomes impossible to distinguish between the disciplines. There are no longer borders between science, technology, engineering, and mathematics. From the first scenario onward, there is a steady progression of integration leading up to the goal of total integration.

Mathison and Freeman (1998) also present five levels of integration for STEM disciplines. The first is intradisciplinary, where the connections within the disciplines are enhanced. An atom’s electrical charge is both a chemistry and a physics concept and mentioning this is an example of intradisciplinary teaching. Next is cross-disciplinary. In this level, coordinated contents across separate disciplines are planned. The emphasis is placed on the transferable skills across the disciplines. The third level is interdisciplinary. Here, one skill, concept or method is viewed using multiple disciplines. The fourth level is integrated where disciplines are lost in a global perspective and the approach is inquiry based and issue oriented. The fifth level is integrative and is also inquiry based but student/teacher negotiated, and issue directed. There is more freedom for student exploration. It becomes clearer that regardless of the model that’s taken, the result seems to reflect the same idea. True integration knows no singular discipline. It becomes a completely homogenous blend of skills and concepts that allows for a better understanding of the problems of the world and gives better means of solving them.

A point to consider is that true integration is complex. Concerns may arise in relation to the structure of the curriculum or the evaluations of its criteria. Two common problems can occur. One is that integrative teaching is done in class activities, but evaluations remain standardized. The second is that the concepts learned are superficial and yield no true utility outside of the classroom (Mathison & Freeman, 1998). Therefore, the path to true integration begins with the role of the curriculum. Beane (1995) suggests that a curriculum should express a problem or an issue. This problem can be a personal problem, or it can be a societal problem. Creating a curriculum in this manner will help students develop a better understanding of themselves and the world around them as well as acquire knowledge in an organic way. It allows for total integration because life does not have the boundaries disciplines do. When presented with an environmental issue like climate change, it becomes impossible to address the problem in terms of a single discipline. To properly address the problem, considering aspects like (a) the effects of aerosols on the ozone (chemistry), (b) renewable energy (physics/engineering), (c) the wildlife of different ecosystems (biology) and (d) the economic effects of reducing fossil fuel usage (mathematics) can all be beneficial. Societal issues are rarely, if ever, a one discipline issue. Beane (1995) also specifies that there is no intermediate step in which disciplines are identified in the problem. The goal is to use knowledge without the labels of the disciplines. The focus is on the activities and projects, not on the subjects. If the activities are thoughtfully constructed, they will inevitably draw on a variety of disciplines. This is not to say that some disciplines are not more apparent in certain societal issues than others. But there lies the beauty of collaborative teaching. Every teacher has multiple areas of expertise. When a person looks at a societal issue, the discipline(s) with which the teacher is most familiar becomes more apparent. The more experts that collaborate, the easier it is to integrate different angles when studying an issue. The key is to avoid separating disciplines once the curriculum is in its final planning stages. Collaborative teaching facilitates this.

Another approach to planning a curriculum is the 11-step model created by Harden (2000). This model relies on collaboration and co-teaching between the different STEM departments (i.e., the physics, biology, chemistry and mathematics departments of a certain university) but could also be used in the school system. The first step is isolation. The subject specialists individually consider their subject matter, what they consider to be important concepts to learn in the classroom. Step 2 is awareness. They learn of the objectives and concepts the others wish to bring to the classroom. Step 3 is harmonization. They consult each other and create connections between the concepts and the objectives. Step 4 is nesting. Here, the specialists will look through their own subject planning and identify skills that could be related to the other subjects now that they are aware of each other's curriculum. Step 5 is temporal co-ordination. All the specialists schedule their teaching programs while consulting each other. Step 6 is sharing. Specialists can plan co-teaching when they find overlap between concepts. Step 7 is correlation. In this step, much of the curriculum is still subject based, but a new dimension of integrative teaching is added. Students will learn the individual subjects first then be presented a lesson of co-teaching where they are integrated. Step 8 is the complementary programme. The integrative lessons are now most of the focus of the teaching. Scheduled opportunities for subject based lessons become a small part of the curriculum and are placed for students to gain a deeper understanding of specific subjects. Step 9 is multi-disciplinary. The borders between disciplines become thinner and thinner. The theme in a programme becomes a structured body of knowledge that transcends disciplines. There is no longer subject-based teaching. Harden (2000) gives the example of the endocrine system since this model was created for the medical field. It becomes the body of knowledge that's needed to be learned and the physics of the body, the biology of its anatomy and physiology and the chemistry of the drugs used are the disciplines that are no longer distinguishable. Step 10 is inter-disciplinary. There is now a loss of the disciplines' perspective. The theme now revolves around a phenomenon where multiple disciplines are needed yet not identified. The final step is trans-disciplinary. In this final step, the focus is now on the knowledge of the real world. There are no subjects to learn. Students are provided a situation, and the integration is done in their mind. This reflects the result of the models proposed previously.

Although the conclusions of the different approaches are similar, elaborating the different steps of integration provides a useful tool. It can sometimes seem like an impossible task to go from no integration to complete integration. But deciding on the scope and level of integration that is realistic in a teacher's specific circumstances is better than no levels of integration at all. Also, these approaches rely heavily on collaboration and individual responsibility. It could be to an organisation's benefit to offer training to staff members to improve the planning, organization, and execution of curriculums (Malik, A. & Malik, R., 2011).

## Where?

As we have shown, the idea of integrated STEM is not new. However, the lack of readily available integrated STEM education is a testament to its complexity in practice. Postsecondary institutions offer a variety of STEM programs. Some could be described as intradisciplinary or cross-disciplinary STEM programs, where departments try to show students the connections between disciplines. You may also find some institutions who offer integrated science programs. They are not STEM, but the "S" is fully integrated. An example of this is the iSci program at McMaster University.

The iSci program is a four-year undergraduate program that offers an innovative pedagogical design and delivery model to better prepare students to face global science related challenges like climate change, pandemics, and renewable energy. As stated by Symons et al. (2017), this model, called Research-based Integrated Education (RIE), allows students to construct their understanding of science in four levels. Students start on level 1 with literature-based research and argument construction and continue to level 2 with developing research questions and original models along with collecting and analysing data. By level 3, they select the research topic they want and put emphasis on science communication until they reach level 4, which is their undergraduate thesis. We can see the integration because the borders between the science disciplines are not present. The program focuses on problem solving and the skills required to do so, mainly the ability to find and organize information and create new meaning while also being able to communicate that meaning.

As for complete STEM integration, there is one program that seems particularly interesting. It is called the iCons program, a four-year undergraduate STEM program offered by the College of Natural Sciences at the University of Massachusetts. It includes a course every year and a year-long thesis. Auerbach (2015) offers a quick synopsis of every year. In year 1, students have a course on global challenges and scientific solutions. They learn about the attitudes and methods of integrated science and practice teamwork from case studies taken from current real-world topics. The final project involves students designing their own case study. In year 2, students have a course on integrative science communication. They are given themes like renewable energy and biomedicine and learn to engage with issues related to the theme through writing, reading, speaking, and debating. In year 3, students are given the opportunity to apply their learnings through team discovery laboratories and present their experiment findings. Finally, in year 4 the students design their own interdisciplinary research project.

This program draws on real-life situations and the disciplines naturally follow. In terms of integration, we see the students start fully integrated in year 1. Once they are in the right mindset and learn the necessary methods, they are given specific scenarios based on themes chosen by the students. The themes themselves show a degree of interdisciplinarity. They can then refine their methods and develop transferable skills through theoretical problem solving and concrete problem solving. Finally, they take their new learning and re-apply what they learned to real-life situations. It takes the approach suggested by Beane (1995) and applies it in both directions. The program starts and ends with real-world issues, which gives the framework for STEM integration and then it develops specific transferable skills in the middle, creating a STEM program almost in the shape of an hourglass. It's a program that we feel truly embodies integrative STEM education.

While these programs are not targeted at educators, they could provide valuable insight into the training of future teachers, as several ideas, including the use of real-life situations to integrate science, technology, engineering, and mathematics, seem particularly interesting for meeting the STEM education needs of individuals enrolled in a Bachelor of Education degree. Given that teachers feel ill-equipped for STEM education, what better way to familiarize them with STEM education than to have them live it? That is what we attempted to do.

## STEM in our didactics courses: Take 1

During the 2021 Fall semester, the mathematics education course as well as the science education course had the same schedule. That simple detail led to a collaboration between both professors of these courses. We co-taught part of the mathematics education and science education courses and engaged students in a collaborative STEM experiential learning project. Because of the special circumstances of COVID-19 such as social distancing and mask wearing, a few of the co-taught classes were given outside on the lawn around the Faculty of Education. The students sat on beach towels or lawn chairs and worked on their tasks. The professors also taught while outside. In October, the students visited an organic farm (figure 1) and transferred their new knowledge to a classroom context by developing a prompt, that is, a problem that could be presented to high school students and lead them to use STEM to suggest a solution. For example, some students looked at the possibility of creating a new kind of tomato, while others were more interested in the entrepreneurial aspect of the farm and selling vegetable baskets. After receiving formative feedback on those prompts from the two didactics professors, the students then had to transfer their knowledge and skills to a larger context, that of developing a month-long STEM unit for a high school class. They had to create a meaningful learning situation for high school students, keeping in mind the achievement of curriculum outcomes, as well as the development of digital skills and the three competencies of the New Brunswick Acadian and Francophone school system's exit profile: social-emotional skills, cognitive skills, and communicative skills (Ministère de l'Éducation et du Développement de la petite enfance, 2016). They discussed this situation with two high school teachers (math and science) in an environment drawing on professional learning communities. This discussion with experienced teachers allowed them to improve their STEM teaching unit to better meet the challenges of the New Brunswick school system. Among other things, teachers raised some time constraints (students often take longer than we initially thought they would). They also stressed the importance of ensuring that learning is meaningful to students, in other words, that the problem being studied is not only realistic, but also close enough to the students' own reality that they can feel compelled to engage.

**Figure 1: Visit at the organic farm**

## Research project potential

We noticed that teacher candidates are particularly active and engaged when working on this type of project, sometimes exceeding our expectations. Although this project was initially intended to be pedagogical, we now recognize the strong research potential to inform a little-known area of university education, namely STEM teacher education for students enrolled in the Bachelor of Education program. Therefore, what began as a pedagogical project was transformed into a research project. This project falls within the qualitative paradigm. We wish to carry out a case study in which we will be interested in both the teaching and learning of STEM in the university context and more specifically in the mathematics didactics course and in the science didactics course. The case we are interested in is the collaboration between two didactics courses that took place in the Fall 2021 semester. More specifically, we aim to:

- Understand how to integrate STEM education into mathematics education courses and science education courses through experiential learning and faculty collaboration;
- Synthesize the higher order skills and abilities developed by students enrolled in mathematics and science didactics courses as they develop STEM teaching situations.

The Université de Moncton, Moncton campus, welcomes approximately 4,000 students annually. Roughly 550 are enrolled in the Bachelor of Education program, and about 200 of them have chosen the secondary school teaching option, a quarter of whom have a major or minor in mathematics or science. If we divide these students almost equally across the five years of the teacher education program, we have very small groups of students in both math and science didactics, which has facilitated the collaboration between the two courses. In the Fall 2021 semester, four students were enrolled in the mathematics education course and three were enrolled in the science education course. We asked these seven students to participate in an interview to help us better understand their experience in the course. Interview questions focused on collaborative sessions between students of both courses (e.g., writing questions to prepare for the farm visit during class time), the visit to the farm, the creation of STEM didactic situations (formative and summative), the participation in a professional learning community with both teachers, and the feedback given by the professors throughout the course. We also asked the students what were the main learnings (didactical and pedagogical) that were achieved and what were the key elements that could be transferred to their future practice. Three of them answered our call positively. A research assistant conducted the semi-structured online interviews with these three individuals. At the time of the interview, the two didactic courses (math and science) had been completed, the grades had been handed in, and students had no further courses with these two professors until the end of their undergraduate studies, eliminating the participants' desire to respond positively to questions to increase their grade in the course. The research assistant transcribed the interviews. During the transcription process, all traces of the participants' identity were removed. Given the small number of students enrolled in the two courses and to respect the anonymity of the individuals, each participant was given a transcript of the interview to decide if he or she wished to have any passages omitted from the analysis, as they provided a means of identification. Doing so allowed us to give participants the opportunity to validate the interview transcript. We then conducted a thematic analysis of the verbatims to highlight the main ideas mentioned during the interviews.

## The students' perspective

Students talked about the integration of STEM education into mathematics education courses and science education courses through experiential learning and faculty collaboration. When asked about their experience, four themes emerged: the invaluable feedback received from the teachers, the farm visit, the appreciation for the courses that were offered outdoors, and the relevance of the feedback from both professors throughout the semester. The feedback from the teachers led students to reflect on things they had not thought of and, as a result, allowed them to improve their work.

Student B: It made me realize things a little bit. Like, in the future, do not forget this when you are doing your planning. Things that I can use for this project, but also for other projects, for other lesson planning.

Student C: We would read the comments and brainstorm improvements. Then we would move on. Then, like at the end, our project really improved.

With respect to the farm visit, some talked about the richness of the visit, while others were left wanting more: "It seems, however, that there were several questions that we were asking ourselves, and then we were hoping to get answers that were a little, how shall I say, more complex, more laborious than what we got from them." Finally, the professors' feedback was appreciated, particularly because it allowed them to improve their work before the final summative evaluation, a process in which they had experienced little during their initial training.

Student A: Often, we are given feedback on what we need to improve, but we were also told what we did well, so that is something that solidifies our learning. […] Often, we do an exam, then once it is done, it is over, it is put aside and you do not come back to it, and it should not necessarily be like that. You should be able to go back and improve your work.

We were also interested in knowing what should be kept and what would benefit from being modified in the collaborative course. The issues raised on this question are presented in Table 1.

**Table 1: Elements to be kept or modified in future versions of the collaborative STEM course between the mathematics education course and the science education course**

| To keep | Improvement |
|---|---|
| Visit to the farm | More flexibility with the visit<br>A greater significance in the final project |
| STEM project | More time to explore different problems<br>Clearer instructions<br>A debriefing on the project<br>Better overall organization |
| Outdoor work | |
| Talking with the teachers | |
| Formative assessment | |
| Long-term work with the same group of peers | |
| Collaboration between two courses (not necessarily combined) | |

In general, the collaboration between the two courses seems to have been appreciated. The elements to be improved lie in the freedom granted to the students (they would have preferred to choose the place they were going to visit, rather than being forced to visit the farm, as not all of them were interested in the issues that emerged from the visit.) and in the general organization of the course (since it was the first time that we were offering these two courses by co-teaching, we recognize that some decisions made during the semester caused some stress for a few students).

Finally, we looked at higher order skills and abilities developed by students enrolled in mathematics and science didactics courses as they develop STEM teaching lessons and units. Three main themes emerged from our analysis: the significance of the work done, their ability to collaborate, and their commitment. Also, the students were able to make connections between what they had learned and their future practice.

> Student B: I am thinking about incorporating situations that are more relevant to what is going on in the world, but also trying to have an openness, trying to leave a little bit more control to the students. Here is what to do, here is a project, here is a task. What do you need to learn to accomplish the task?

They also recognize the power of collaboration, both with their peers and with their professors, in their learning.

Student A: It was really a work environment and not a professor-student environment. It was really a collaborative work environment, which was different from my other classes.

Student C: Especially in education, I like to collaborate or do team projects because that is how you grow, compared to doing projects alone.

The important level of commitment, which was reflected in the quality of work delivered and the exceeding of expectations set in both courses, was noticed by the professors, but more importantly by the students.

Student B: I saw far more how it could help me in my future profession. I saw way more how it could help me in my classroom, what I could use in my classroom, then I liked it. It engaged me to put in more effort, then put in more time than other university courses.

Student C: In all my didactics, I have been proud of my projects, but I think STEM was really the "pinnacle," yeah.

A particular example caught our attention. One team presented a four-week learning situation that could be done with grade 9-12 students (when they only had to target one grade). They integrated not only the elements seen in the mathematics education course and the science education course, but also some seen in other courses in their university training (for example, the importance of involving students in the development of evaluation criteria, a principle seen in the course on assessment). In addition, they were able to identify several interesting documents with authentic data, as well as a variety of experts who could visit the students to support their learning.

## STEM in our didactics courses: Take 2

Considering the feedback received by our students, we decided to retain certain elements and improve or modify other elements in a second version of this course for the Fall 2022 semester. Some of the elements we will be retaining are the STEM project, the outdoor work, the professional learning community with high school math and science teachers, the formative assessments, the co-teaching of certain classes, and the long-term work with the same group of peers. In this second version, the STEM project will consist in the design of an actual outdoor classroom. The students will use the engineering design process to create a model of an outdoor classroom that could be built on the campus of our university. In the beginning steps of the design process, the students can choose to visit a school in New-Brunswick that has an existing outdoor classroom and collect data and ask questions. Then they can integrate elements of what they liked and found relevant in their own design. They will present their ideas to their peers and the professors and receive feedback. They will subsequently create a STEM unit for high school pupils that integrates their outdoor classroom design and present their ideas to the high school teachers participating in the professional learning community and receive their feedback. Once again, they will have a chance to improve their STEM unit before the final summative assessment. We hope to offer the students more flexibility in the visit for them to decide and coordinate when and where they visit the location and to make feedback they will receive during the professional learning community more pertinent to their STEM project and their STEM unit planning. We will also clarify the instructions in the documents that describe the tasks and improve on the rubrics for assessment. We wish to better organize the group work and the individual work. It is also important to note that the two professors received a research grant from the

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# Beyond flipped classrooms: Students' learning experiences in an undergraduate physics course

Dominic Manuel[1] and Marc de Montigny[1]

*We present a collaborative pilot study between three professors at the Campus Saint-Jean, University of Alberta, in which we examined students' learning experiences when the instructor used two student-centered teaching approaches: inquiry-based learning (IBL) and flipped classrooms in an undergraduate physics course. Inspired by Marshall et al.'s (2009) 4E × 2 inquiry-based teaching model and Lebrun and Lecoq's (2016) three types of flipped classrooms, we designed and enacted activities that combined the two approaches throughout the Fall 2021 semester. We collected participants' reflections after each activity and conducted semi-structured interviews via Zoom with four participants at the end of the semester. Preliminary results from a thematic analysis revealed promising learning experiences with these approaches for learning, mostly with IBL. However, students felt that expectations towards the videos were not always clear.*

*Keywords: Inquiry-based learning, flipped classroom, science education, university physics course, student experiences.*

## Context of the study

The complexity that persists in all spheres of society as well as the exponential growth of new digital technologies that emerge in all sectors require important changes in the way citizens think and reason mathematically and scientifically, as most often, these competencies can go beyond what is taught in classrooms (Lappen, 2000). It thus becomes more essential to better educate students both in schools and in postsecondary institutions in the science, technology, engineering and mathematics (STEM) fields by supporting them in developing such competencies, such as critical and creative thinking and decision-making in science, and use technologies efficiently so they can adapt themselves to the reality of a society continuously changing, and become vectors of progress and leaders in their respective fields. This thus elevates teachers' expectations regarding teaching mathematics and sciences, as they must implement high-quality teaching practices to create effective learning conditions in classrooms (Ball & Bass, 2000; Ball & Cohen, 1999). For many decades, researchers and educational administrators have emphasized the importance of developing various competencies in students, such as problem-solving skills, inventing and creating their own strategies and experimentations, defining and verifying hypothesis, manipulating variables, collaborating in groups and interacting with colleagues, and communicating facts and proofs (Lesh & Zawojewski, 2007; O'Connor & Michaels, 1993; Windschitl, 2003; Windschitl et al., 2012). Many researchers in mathematics and science education propose using more student-centered approaches in classrooms in which students take more responsibilities in their learning process by engaging themselves in

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various tasks and interacting with their peers and the professor, as together they co-construct new knowledge (Mass & Artigue, 2013) This includes the sciences such as physics.

Physics is the most fundamental of natural sciences; it deals with the smallest phenomena such as the atomic nucleus, as well as the largest structures such as the universe and cosmology. Observing phenomena and studying the relationships between variables are part of important responsibilities of physicists so they can make new discoveries to better understand our universe, to predict problematic situations that occur and that could potentially influence the future, and to use critical thinking and decision-making skills to find creative and innovative solutions to those problems. Moreover, many discoveries in physics are made because of the important role that technologies play in the laboratories. We thus suggest that doing science requires students to be engaged in ways that allow them to work more like scientists and still learn theoretical and practical foundations related to physics in classrooms. Most of these competencies that physicists use are similar to those essential for citizens to adapt and be able to contribute to their society.

In addition, pedagogy in physics is also a developed subject because of its versatility and the fact that many phenomena can be understood by means of various representations, such as equations, graphs, models, demonstrations, experiments, technological simulations, and others. These representations naturally permit the use of different models when learning and can not only support students in developing deeper meaning of concepts taught, making links with other concepts. In addition, these representations may adapt to various students' learning styles. In addition, these representations are possible because of the role that technologies can play in the learning process. We thus argue that the growth of technologies in our society nowadays can not only influence *what* is learned in classrooms but also *how* we can learn and discover new content. We also suggest that technological resources must be an integral part of the learning process of future scientists as well as for future teachers who will educate the future generations.

Unfortunately, instruction in sciences in schools and in postsecondary institutions remain direct or, in other words, teacher centered (Maass & Artigue, 2013; Aulls & Shore, 2008). The professor takes the responsibility to transmit the information to the students, and students apply the knowledge via practice exercises often done at home. In their meta-analysis of studies on the implementation of IBL in mathematics and science classrooms Bruder and Prescott (2013) suggest that although there appears to be strong evidence on the positive effects of IBL on student learning, very little attention is put on the implementation process, which could partly explain why this approach is rarely used in classrooms. We suggest that this teaching model harms the training of all learners. First, this model does not support future scientists to develop the necessary competencies that scientists need in their careers. They simply learn content in their courses. Second, future secondary teachers do not get opportunities to do science and develop a scientific culture in their program when they take content courses. How can future teachers develop the necessary competencies in creating student-centered learning conditions in classrooms when they cannot experiment with science as leaners in their content courses? Yet, student-centered approaches are recommended in the science teaching reforms for many decades (National Research Council, 1996, 2000, 2011). We think that the pilot study presented in this paper could give some insights on the implementation process of IBL in physics.

In this pilot study, which is a collaboration between a physics professor, de Montigny, and two professors in education: one specializing in mathematics and science education, Manuel, and one who studies the use of technology, Pellerin, we aimed to improve the students' learning experience in a physics course taught at the Faculté Saint-Jean, University of Alberta. Most students who take this course are enrolled in the secondary education program offered at the Saint-Jean campus, while others are enrolled in the science program offered at the same faculty. To improve the quality of teaching, we implemented a combination of two teaching and learning approaches that are considered as student-centered approaches: inquiry-based learning (IBL) and flipped classrooms. In general, IBL is an approach through which students learn concepts and acquire knowledge in ways that mirror the work of scientists while flipped classrooms is generally an approach where technologies facilitate the learning process by dividing what is learned in both inside and outside the classroom. In both these approaches, instead of simply being receptors of information transmitted by a professor, learners take more responsibilities in their learning process by exploring various interrogations and by constructing their knowledge by means of interactions with their peers and their professor. These approaches are thus the opposite of traditional instruction, which is commonly used in university undergraduate science courses. We are attempting to enrich the education of both future scientists and teachers who study in francophone minority institutions in Canada, such as the Faculté Saint-Jean.

## About the Faculté Saint-Jean

The Faculté Saint-Jean is a francophone postsecondary institution that is part of the University of Alberta, in Edmonton. It is home to approximately 750 students, including about 400 in education programs and 220 in sciences. About 70% of the student body originates from French immersion programs. Therefore, learning in a second language is an additional difficulty faced by our students. This being said, we did not specifically investigate the language component in our study. We must add that students registered in the secondary education program and majoring in science only have one science education course. We thus believe that this project could also serve as a "didactic laboratory" (science education laboratory) where students are exposed to the scientific culture and explore as learners what it means to do science. Moreover, since IBL and flipped classrooms both facilitate a construction of knowledge between the professor and the students by sharing roles and responsibilities in the learning process, and an emulation of explorations similar to those scientists use, we suggest that this experience will also benefit the students enrolled in the science program as they will get opportunities to work in ways that will be demanded in their future careers.

## Goal and research question

Research conducted on IBL in science classrooms has revealed that this approach is effective for fostering deeper learning opportunities (Aulls & Shore, 2008; Bruder & Prescott, 2013). In a similar vain, flipped classrooms are associated with higher academic achievements (Chen & Yeng, 2019; Torio, 2019). As far as we know, little is known about postsecondary students' learning experiences with flipped classrooms, or with a combination of IBL and flipped classroom. Our goal is thus to examine students' learning experiences of undergraduate students enrolled in a physics course (PHYSQ 124) that we describe in the methods section where both IBL and flipped classrooms are implemented. Our research question is: How do undergraduate students enrolled in the PHYSQ 124—*Particle and Waves* perceive their learning experiences when IBL and flipped classrooms are used as teaching methods? We assume that the combination of both approaches may reveal traces of a possible theoretical framework that can inform approaches beyond flipped classrooms.

## Theoretical framework

In this section, we describe the theoretical foundations for the two pedagogical approaches: IBL and flipped classrooms, on which we based our activities in our projects. For each approach, we present a working definition and a model, which guided us throughout the project.

Inquiry-based learning is defined as a student-centered teaching approach in which students, working individually or in small groups, develop disciplinary and interdisciplinary knowledge and competencies by working in ways similar to those of scientists (Chickekian et al., 2011). Inquiry-based learning is considered a socio-constructivist learning approach, as the teacher and students share the roles and responsibilities in the learning process (Aulls & Shore, 2008), and co-construct knowledge by means of interactions (Manuel, 2020). Moreover, IBL is also known as the "preferred teaching approach" to foster deep conceptual learning (Aulls & Shore, 2008), and to develop students' interests towards science, technology, engineering, and mathematics (STEM) fields (Rocard et al., 2007).

Marshall et al. (2009) proposed the 4E × 2 teaching model for teaching mathematics and science using IBL. This model (Figure 1) contains three main components that teachers must consider while using IBL in classrooms: assessment, reflection, and an instructional approach consisting of four phases: Engage, Explore, Explain, and Extend.

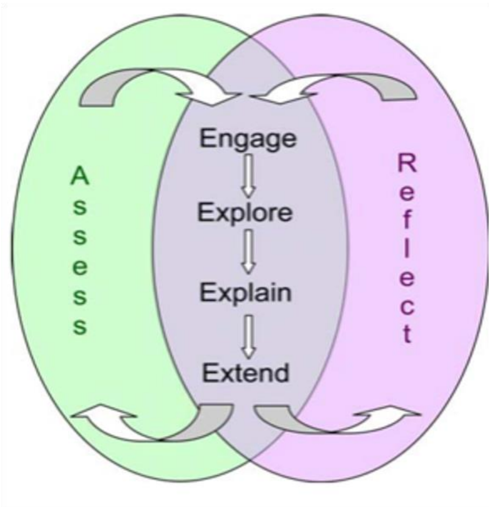


**Figure 1: 4E x 2 IBL instructional model (Marshall, 2009)**

The Engage phase focuses on probing prior knowledge, identifying alternative conceptions, providing motivating and interest inducing stimuli, and developing scientific questioning. Students are usually exposed to an interrogation (task) in this phase. In the Explore phase, students work on the interrogation by predicting, designing, testing, collecting data, and reasoning. The Explain phase is reserved for interpreting data and findings, providing evidence for claims as well as communicating findings and providing alternative explanations for findings. In the Extend phase, teachers and students apply, elaborate, transfer, and generalize knowledge to novel situations. According to Marshall et al. (2009), the four phases are not necessarily linear, as some can be repeated in the process. For example, in the Explain phase, students might come up with other questions that could lead to another exploration (Explore phase). However, the authors also stress that the Explore phase must precede the Explain phase for instruction to be considered as IBL. In fact, the Explain phase before the Explore phase is considered traditional teaching according to the authors.

Flipped classrooms were initially defined as a teaching approach in which students learned content at home with videos or readings, and did coursework in class (Bergmann & Sam, 2009). The typical elements of a course are reversed, both in time and in space. We were much influenced by Lebrun and Lecoq (2016) who further saw flipped teaching as a setting where students play a more active role in the development of knowledge, and teachers and students thus share their roles in the learning process. Lebrun and Lecoq (2016) described three types of flipped learning (Figure 2). In *Type 1*, the "knowledge" is externalized via digital technology in an autonomous way with activities to accompany the learning in class. *Type 2* is the inverse. Students research knowledge by exploring a theme independently (or in groups) and present/discuss/debate their conclusions in the classroom, thus sharing or changing roles between teacher and students. *Type 3* is a combination of Types 1 and 2 by alternating contextualization, decontextualization, and re-contextualization activities.

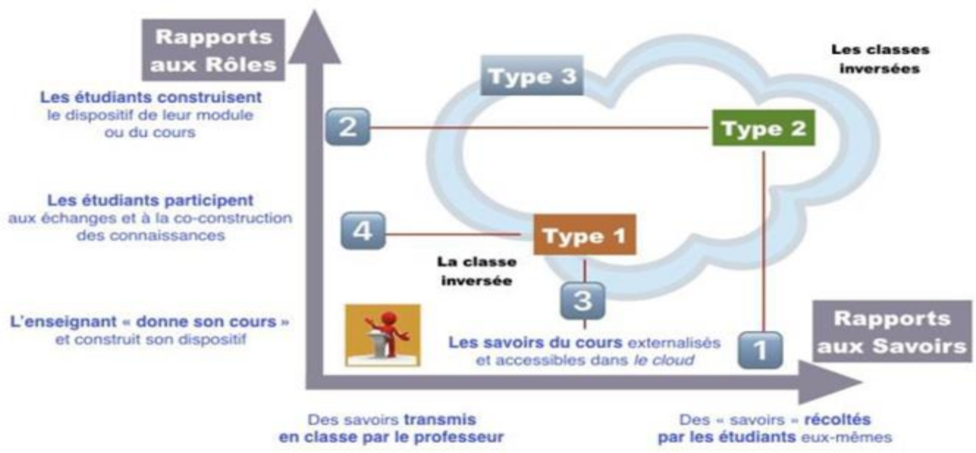


**Figure 2: Three types of flipped classrooms (Lebrun & Lecoq, 2016)**

We suggest that we go beyond flipped classrooms by combining this approach with IBL.

## Method

We followed a qualitative interpretative method (Karrsenti & Savoie-Zajc, 2018) for our study. We found that collecting qualitative data would permit us to get a better grasp of the students' experiences.

We used combinations of the two approaches by designing and enacting activities in the physics course PHYSQ 124 called Particles and waves, taught by de Montigny at the Faculté Saint-Jean during the Fall 2021 semester. This introductory course covers kinematics, dynamics (Newton's laws, energy, momentum), conservation principles, rotational motion, oscillations, waves, sound, and light and photons. Fifteen students were enrolled in the course, eight of whom were in the secondary education program. Twelve students participated in our study. The PHYSQ 124 course also includes a laboratory session, but we conducted our research only on the theory part of the course. The course was taught in person for part of the semester. However, due to the COVID-19 pandemic, some classes were taught online on Zoom around mid-semester. The professor agreed to teach the rest of the semester bimodally, that is, both online and in person at the same time with some students attending the class in person and others on Zoom. Most of the classes were video recorded. Classes were 80 minutes long, and they were held twice a week for a period of 13 weeks.

The activities were designed using the 4E × 2 and the three types of flipped classrooms models. Each week, we, the research team, would meet online at Zoom for approximately two hours. During the sessions, the professor would present initial ideas of activities and other members of the research team would validate and give feedback on the design of the activity as well as important teaching practices to implement while enacting those activities. We made sure that all the activities respected an IBL model of teaching and had components for flipped classrooms. We also formulated questions to ask the students as well as other important teaching practices that the professor should use while enacting the activities. We took time to debrief on the previous week's lectures in order to improve classroom experience if necessary. Lamiah Fahim, our research assistant (RA) would give us feedback based on the participants' reflections. Most of our sessions were video recorded as well.

As for the activities created, most of them contained a combination of IBL and the use of technologies, mostly videos and online simulators. Some parts of the activities were done at home, while other parts were done in class. For instance, in one activity on projectiles, we followed mostly a type-1 flipped classroom activity. Students watched at home prior to class a video on the different equations that can be used in kinematics, mostly on when gravitational acceleration had an impact on the movement. Then in class, they were exposed to an IBL problem. Students had to know which equations would be helpful to find a solution to the IBL problem. In another activity on Hooke's law, students began the activity in class with an IBL in which they used a simulator to investigate the relationship between the force by the spring and the distance from the point of equilibrium, and then completed the activity at home watching a video on the topic.

We used reflection forms and semi-structured interviews as data collecting tools. Each week, the participants would complete a reflection form using Google Form about their experience with that week's activities. This would be managed by our RA, the research assistant. At the end of the semester, our RA conducted audio-recorded semi-structured individual interviews on Zoom with four participants. She transcribed the interviews and gathered and summarized all the weekly reflections. We must add that our RA was responsible for recruiting the participants at the beginning of this study and for collecting all the data because we needed to be certain that the professor of the course, who is part of the research team, could not identify the students participating in the project.

We used a thematic analysis (Braun & Clarke, 2006). We identified elements that emerged from the corpus of the interviews as well as the reflections. We then categorized the elements in appropriate themes. We present our preliminary results in the next section.

## Preliminary results

Four themes emerged from the data. The first one, which was a central topic of discussion for the participants during the interviews, was the learning potential of the approaches. Students found that the activities they explored in class and at home provided them with opportunities for more dynamic learning experiences. They also added that the activities pushed them to be more autonomous in their learning experience and to experiment ideas before being taught the content in class. "I loved both pedagogy…I think that it is always good to try out new things on our own before they teach us the methods: how to do the calculations…" (Participant 1, interviews). Interestingly, students seemed to favour the IBL approach over flipped classrooms, partly because the IBL activities (in particular, the Engage and Explore phases) established links with their real-life experience. "I really preferred IBL because we had a model to follow with questions that guided our learning process" (Participant 3, interviews). Participant 2 explicitly mentioned during the interviews that "IBL helps us make links with the real life." It is to be noted that most of the content covered in the course was concepts that students have learned in their secondary physics class. "Since the content we learned this semester resembles somewhat to what we learned in Physics 20 in high school, I knew the content. But I was able to make links between the content and real-life experiences" (Participant 2, interviews).

The second theme was intrinsically related to the first one as it pertained to the actual delivery of the various activities. This theme is the availability of resources that accompany such activities. Students pointed out that the availability of multiple learning approaches and multiple resources rendered the course more engaging and dynamic. For instance, participant 1 mentioned that "The resources helped me explore the content in various angles besides simply listening to a presentation or listening to videos" (Participant 1, interviews). Participants also pointed out that the resources supported students in making the content more concrete. "The questions that we had in the separate documents really made the content more real like concrete. There were a lot of examples. I really appreciated the simulation with the hooks" (Participant 1, interviews). In addition, some students acknowledged that the videos used with the Type-1 flipped classrooms were good learning guides. In fact, in their reflection form, participant 6 mentioned that "when I didn't understand something, I could go and listen to the video and listen to it again if needed…. You can accelerate or slow down the videos to help you understand. The Prof is gone after (class is over)."

The third and fourth themes that emerged revealed certain challenges that occurred while implementing IBL and flipped classrooms in the physics course. As the third theme, students discussed the necessity to become more engaged in their learning and to also often interact with other classmates. The participants mentioned that they had to be more engaged in the learning process, and that they had to interact with the other classmates during the classroom activities. It appeared that this was quite new for students, and it required some adjustments during the semester.

> At first, I didn't like flipped classrooms. It gave me stress because I had to listen to videos or prepare myself at home. I had a lot to do at home, and the course was pretty packed. But as the semester went on, I warmed up a bit. I liked the fact that we could access most of the resources long before the class… At the end, I appreciated the fact that the Prof forced us to view some materials before class. I became more at ease with the course" (Participant 2, interviews).

However, as the semester progressed, it appeared that the adjustments brought more self-confidence in students. "I got my confidence. It went up because with others, we had more chances to better learn the content" (Participant 1, interviews).

The last theme that emerged focused on the expectations of the videos as a learning experience or support. Some students claimed that they were unsure about the expectations and the roles of the videos, especially when a Type 1 flipped classroom was used. They did not feel that watching the videos was compulsory, as the instructor would tend to repeat the theory content of some videos, apparently rendering their viewing redundant. Some participants even wondered whether the videos were a differentiation practice or just a review. Based on the way the classes went, one student admitted that watching the videos was not really necessary. According to Participant 2, “The videos were a good introduction, but I felt I understood the content more when the Prof explained it in class after” (Participant 2, interviews). In addition, some students also felt that the videos were not sufficient to support their learning. Participant 4 explicitly mentioned during the interview that the videos were not enough to understand all the content and be ready for a test or a quiz.

## Concluding remarks

The preliminary results of this pilot study seem promising to improve the quality of the learning experience of learners. In general, the participants seemed to have appreciated the two approaches introduced in the physics course, particularly IBL. It also appears that some students in the education program might have already made links with teaching secondary science. In fact, the students seemed to have observed that using a combination of IBL and flipped classrooms brought a different dynamic in the classroom, one that pushed them to be more engaged in more autonomous ways as well as to learn to interact and co-construct knowledge with other classmates. Overall, the students’ experiences seemed to align with the motivations discussed in Lebrun and Lecoq (2016) for flipping classrooms.

However, as we mentioned, this paradigm change seemed to have created a challenge for the learners from the start. Some participants clearly mentioned that it took time for them to adjust to the change in classroom (and outside of the classroom) activities. This experience seemed to them to be something new and unexpected. This aspect relates to Brousseau’s (1998) concept of didactic contract. This contract consists of all the implicit and explicit relations and conditions that are established between the professor and the students about the learning. In this project, de Montigny modified the didactic contract by integrating approaches that were possibly not known for students, or that students were not used to. This change in the didactic contract thus created uncertainties among the students when it came to their learning, as they were mostly used to having their professors present the content.

In addition, the participants also made clear that using these approaches also brought multiple resources such as videos, simulators, etc. We suggest that students saw the benefits of technologies as a support for learning, as they permitted them to view the content in different ways and see various practical and real-life examples of the physics concepts. We must mention that at the end of the semester, one student emailed the professor and asked for his permission to use some of the resources that were used in the PHYSQ 124 course in their future classrooms.

The results of this study also revealed that it is important for the professor to clearly identify the expectations of students when it comes to their learning for these approaches to be implemented with success. In our study, the expectations were not as clear for the role of the videos in the learning process. Some students did not quite see its place in the activities. Some saw it as differentiation of learning or as reviews. This aspect made us reflect on teaching physics as well as other subjects. Our reflections highlighted that it is thus important that professors implement high-quality teaching practices in order to support student learning, not simply having good questions for students to explore, but also having clear expectations of every aspect of the learning process. The semester created opportunities for de Montigny to reflect on his teaching practices in class, for instance, how to elicit students’ thinking in ways that it will permit them to reflect deeper and explain their reasoning, as well as to how to maintain the cognitive demand of the tasks in ways that he does not say too much to students so they could be more creative and use their own reasoning. This latter teaching practice, as well as setting and maintaining expectations are deemed essential and crucial to implement in future teachings so that the videos remain a learning need and not a revision resource. It became obvious that the videos were not always a learning need since the professor reviewed the content presented in the videos, thus lowering the cognitive demand. Students did not need to push themselves further, as they knew that the professor would discuss the content. This is also another example of an explicit didactic contract that was created. We believe that this is an important aspect to consider in the implementation of IBL in science classrooms. More research is needed to bring more details about the implementation process.

## Limitations and future studies

Two major limitations influenced our project. One limitation of this project is that it focused on the perceptions of students' experiences in learning. More studies would be needed, hopefully with a larger sample, to make more precise conclusions on students' learning experiences. Another drawback was that most of the contents covered in the course were concepts which students had already learned while in high school, thereby raising the question as to whether the experience would be different in a course with only new contents. It is possible that the learning experiences were positive because of that aspect, and these experiences would be different if the contents were always new to the students. In that respect, we point out that most of the new topics, not seen in earlier programs, were covered by using the IBL approach.

Along those lines, a potential avenue for further research would involve deepening the learning experiences by assessing the pedagogical approaches when applied to new contents that students have not learned previously. At the Faculté Saint-Jean, this could be done with the second physics course PHYSQ 126—Fluids, fields, and radiation, which comprises newer content than PHYSQ 124.

Another, very interesting, next step would be the laboratory portion of the course. This would enlarge the possible activities considerably. For instance, the Engage and Explore phases of the 4E × 2 approach to IBL could be accomplished in the laboratory. The laboratory would also allow us to bring further pedagogical approaches. Ultimately, this pilot project could lead to studies in other physics courses and other science courses, as well as other content areas.

## Acknowledgment

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We would also like to thank Martine Pellerin, who was the other member of the research team. We appreciate her comments and feedback throughout the process.

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# Empower students' mathematics problem-solving skills: The role of Computational Thinking

Yimei Zhang[1] , Tanya Chichekian[2] and Annie Savard[1]

*Integrating Computational Thinking into curriculum will foster students' use of their cognitive abilities, which will contribute to their development of problem-solving skills and finally to meaningful learning. In order to investigate the role of CT in mathematics teaching and learning, a case study was conducted to analyze a famous ancient-Chinese mathematics problem, called "Chicken and rabbit in the same cage." In this case study, we define the role of CT in mathematics teaching and learning and explain the connections between CT and problem solving in mathematics and everyday life. The findings suggest that the ability to embrace CT is crucial for mathematics problem-solving, especially when planning to teach problem-solving. Consequently, applying CT into mathematics education could potentially facilitate the transfer of specific skills to other disciplines (e.g., literacy, science, arts, etc.).*

*Keywords: Problem-solving skills, unplugged Computational Thinking, elementary mathematics.*

## Introduction

Problem-solving is one of the prominent skills that fosters students' understanding and application of mathematical concepts and content knowledge (Hiebert et al., 1996). Although many students often report experiencing difficulties with problem-solving in mathematics (Berch & Mazzocco, 2007), due to its importance in daily life and understanding the world around us (Tambychik & Meerah, 2010), empowering students with problem-solving skills is and remains an important goal in education. Computational Thinking (CT) represents a universally applicable skillset. CT skills include managing information effectively and efficiently with or without digital technologies (Wing, 2010). CT has been linked to creativity and innovation, and it has important application in STEM areas. Thinking processes involved in CT refer to formulating problems and deriving solutions similar to those carried out by an information-processing agent (Wing, 2011). Therefore, CT is not only for computer scientists who write the codes, but for anyone operating within digital environments (Wing, 2011). Research in mathematics education shows that implementing CT into the curriculum fosters students' use of their cognitive abilities, and contributes to the development of problem-solving skills, and, ultimately, to meaningful learning (Weintrop et al., 2016). Moreover, the process of problem-solving in mathematics is also conducive to the application of CT (Hambrusch et al., 2009; Jona et al., 2014). This study explores current mathematics curriculums at the elementary level (especially in the cultivation of mathematics problem-solving skills) to find out the potency and traces of application of and awareness to CT in current mathematics education. Specially, this research explores how

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Computational Thinking integrated with mathematics problem-solving (Weintrop et al., 2016). This paper showcases a case study (Lambert & Lambert, 2012), which defines the role of CT in mathematics teaching and learning and highlights the connections between CT and problem solving in mathematics and everyday life.

## Theoretical background

## Computational Thinking

Computational Thinking has been referred to as "a key 21st century skill" and is a vital component for everyone's learning (Yadav et al., 2016). The Computer Science Teachers Association (CSTA) asserts that "the study of Computational Thinking enables all students to better conceptualize, analyze, and solve complex problems by selecting and applying appropriate strategies and tools, both virtually and in the real world." In this research, CT includes the implementation of the following steps: "describe a problem," "identify the important details needed to solve the problem," "break the problem down into small, logical steps," "use steps to create algorithm that solves the problem," "finally evaluate the solving process" (Shute et al., 2017).

In the study, our model is focused on underlying conceptual foundations acquired to solve problems via a CT perspective and clarifying how different dimensions of CT can be highlighted and used in mathematics problem-solving procedures. This is consistent with models that focus on approaching problems, such as Shute et al. (2017), who expand their model of CT to all K-12 subjects. They have categorized CT into 6 facets: decomposition, abstraction, algorithm, debugging, iteration, and generalization.

- **Decomposition**: Dissect a complex problem/system into manageable parts. The divided parts are not random pieces, but functional elements that collectively comprise the whole system / problem.
- **Abstraction**: Extract the essence of a (complex) system. Abstraction has three sub-categories: data collection and analysis, pattern recognition, and modelling.
- **Algorithms**: Design logical and ordered instructions for rendering a solution to a problem.
- **Debugging**: Detect and identify errors, and then fix the errors when a solution does not work as it should.
- **Iteration**: Repeat the design process to refine solutions, until the ideal result is achieved.
- **Generalization**: Transfer CT skills to a wild range of situations/domains to solve problems effectively and efficiently.

Drawing from the aforementioned model and definitions, we defined CT as the conceptual foundation required to solve problems effectively and efficiently. Our model attempts to understand the cognitive processes underlying each of these CT facets. While all of these terms are important in CT, iteration and algorithms are not necessarily required when building a model for CT in our research. Therefore, we only focus on four components in our case study: decomposition, abstraction, debugging, and generalization. Table 1 details the CT model we used for mathematics problem-solving.

**Table 1: Mathematics problem-solving with CT**

| Categories of CT | Sub-categories* | Examples |
|---|---|---|
| **Decomposition** | Read and understand problems | The core of the mathematics problems |
| | | Stages to solve the problems |
| | | What kind of questions are there? |
| | | How are the problems delivered? (Linguistics or Non-linguistics) |
| | Analyze and plan | Break down questions into simple steps |
| | | Break down complex questions into simple stages but identify the same questions that can be solved in the same ways |
| **Abstraction** | Reduction | Remove unnecessary information |
| | | Transfer the problems into a mathematics sentence |
| | Decision making | Decide what information should be included or reduced |
| | Relationship | Identify mathematics relationships within diverse contexts |
| **Generalization** | Pattern | Identify pattern similarities in problem-solving |
| | | Create special models to the problems |
| | Transformation | Transfer ideas from one problem to another |
| | | Conclude characteristics and usage of similar patterns |
| **Debugging** | Confirm the answer and process | Assess that the result is right |
| | | Assess whether the algorithmic solution is good / efficient enough |
| | Test the model and solution | Solve complex questions |
| | | Apply the solutions/process in daily life |

**Note that the sub-categories and Examples & Explanations of mathematics problem-solving are task specific.*

- **Problem Decomposition** is a way to think about problems, algorithms, and processes in terms of their component parts. The parts can then be understood, solved, developed, and evaluated separately. This makes complex problems easier to solve, novel situations better understood, and large systems easier to design.

*Make Cakes* [3]

1. *Bake cake*
    - *Put ingredients in bowl (butter, sugar, egg, flour)*
    - *Mix*
    - *Pour into baking pan*

[3] *(Tasks taken from https://www.csunplugged.org/en/computational-thinking/).*

- *Put in oven for 30 mins*
- *Take out of baking pan*

2. *Make icing*
3. *Put on cake*

Note that the overall tasks of making a cake can be divided into several small tasks, each of which can be performed easily. For the cake-making problem, we could ask several questions when considering different steps: 1) What is the fastest way for making a cake? 2) How much (money) will they spend on making one or more cake(s)?

- **Abstraction** is a way to make problems or systems easier to think about by removing unnecessary information. A key part of it is in choosing a good representation of a system. Different representations make different things easy to do.

*When using digital devices, we use abstraction all the time; these devices will hide as much unnecessary information as possible. For example, when you are asked to plan the fastest way from your house to the university, and now you want to drive (but avoid highways). The digital maps will show you a simplified version of the route by leaving out unnecessary details, such as where every individual tree in a park is, and only keeping the most relevant information the map reader will need, such as roads and street names*

- **Generalization** is associated with identifying patterns, similarities, and connections, and exploiting those features. It is a way of quickly solving new problems based on previous solutions to problems and building on prior experience. Asking questions such as "Is this like a problem I've already solved?" and "How is it different?" is important here, as is the process of recognizing patterns both in the data being used and the processes/strategies. Most importantly, the pedagogical power of computational generalization comes not just from guiding students to use existing models/solutions, but also from enabling students to design, build, and assess models/solutions of their own.

*Pattern tasks are important examples of generalization. When you think about the problems, we might recognize the similarities and connections between them, and the problems can be solved in similar ways. For example, you are required to find the patterns in the graphs and fill in the empty boxes* (Figure 1).[4]

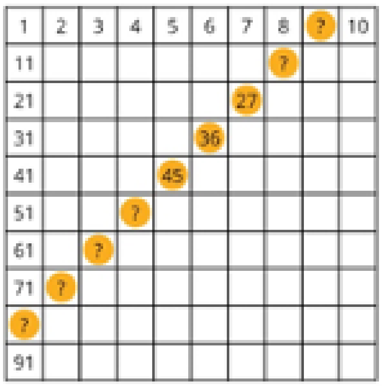


**Figure 1: Pattern in the graphs**

[4] (Tasks taken from: https://www.ontario.ca/page/ministry-education).

- **Debugging** is the systematic application of analysis and evaluation using skills such as testing, tracing, and logical thinking to predict and verify outcomes.

*For example, when your coffee machine does not work and you don't know the reasons, you should detect and identify the errors of the coffee machine to fix it (Task taken from one of the researchers in our group.)*

## Computational Thinking skills & mathematical/everyday problem solving

In elementary school, students are taught to solve problems accurately and effectively in a variety of contexts or to advance from simple to more complex problem-solving (Kenedi et al., 2019). For example, when solving financial problems in mathematics, the students are sometimes required to make reasonable decisions, including how to spend the least money and finally carry out the solutions (Căprioară et al., 2020). The teachers will present different strategies towards one question and connect the question to real-life situations (e.g., what is the relationship between the questions and our real-life problems? What kinds of questions in real life can use this kind of mathematics thinking?), thus cultivating students' efficiency/accuracy competency in mathematics problem-solving (Arthur et al., 2018). However, for students, questions such as how to define the types of mathematics problems (e.g., addition word problems, mixed operations word problems, comparing and sequencing word problems, physical measurement word problems), how to choose the method, when to use the method, and how to define different methods are confusing, especially when confronted with complex questions (Tambychik & Meerah, 2010). Therefore, to raise students' awareness of numerous methodologies and algorithms in problem solving, the importance of abstraction, decomposition, and pattern recognition should be demonstrated in the problem-solving process. These practices are consistent with CT's definitions as described previously.

Problem solving acts as a bridge between mathematics and the real world by enabling mathematics problem-solving within or outside of classrooms or within diverse contexts (e.g., financial contexts, literacy contexts, coding contexts, etc.) to address a variety of problems that everyday experiences may present to us (Boiler, 1993). One of the main goals of mathematics education is to enable students to convert everyday problems into mathematical ones or vice-versa (Fuchs et al., 2004), which can also be regarded as a way of assessing their problem-solving abilities (Carraher et al., 1985). These everyday math problems can be simple, like adding up coins, keeping track of the hours, or figuring out how much something costs. Math is used in everyday life to make things easier and more manageable (Kang et al., 2020).

For example, here is a financial problem: *Betty bought toys for $2.00 each. If she buys 5 toys, how much money will Betty spend in total?*

Answer: *$10.00 - She will spend $2.00 5 times for the toys and that's $10.00 for all the toys.*

In this study, we used the problem "Chicken and rabbit in the same cage," one of the well-known and amusing problems from an ancient Chinese mathematics book "Sun Zi's Mathematical Manual" (Lam & Ang, 2004). In solving problems, we aimed to let students comprehend the entire process of problem solving by using the different CT skills. This problem-solving process can also be used to make a preliminary refinement of objects and then highlight changes in quantitative differences through various contexts or situations to refine a simple problem-solving pattern. The pattern can finally be deduced into a variety of real-life events and problem scenarios, to encourage further internalization of the pattern and complete its creation and application. We must offer students adequate time and space during the teaching and learning process, so that they can explore the problem-solving process while developing a deeper comprehension of this type of problem called "Chicken and rabbit in the same cage."

## Research questions

In the study, we tried to understand how CT was integrated into elementary-level mathematics tasks in order to investigate its role in the developing of mathematics problem-solving abilities and investigate how CT's practical application can improve mathematics education. Thus, there are two research questions:

**RQ1)** How can CT skills be employed to facilitate students' understanding of mathematical concepts and procedures?

**RQ2)** Which CT skills can promote deep mathematics teaching and learning?

## Methodology

The researchers at a university at south-east in China and several in-service teachers in public elementary schools in China joined together in a research-practice partnership to iteratively develop and implement the CT integration curriculum in elementary mathematics. The whole project is design-based research, which aims to help elementary teachers in all subjects (e.g., mathematics, science, Chinese literacy, English) to understand and integrate CT into their teaching. In this paper, we describe the first stage of the project. We limit our report to $4^{th}$ grade mathematics and implement the first round of designing the curriculums—the CT-integration $4^{th}$ mathematics activities. Thus, we analyzed the activities and notes provided by the teachers who designed the activities. For each section of the design, the researchers took detailed notes about teachers' understanding and goals of CT integration.

These data were analyzed through content analysis according to the framework we designed for the research. One researcher assigned the initial codes to excerpts of text that pertained to contents by teachers to integrate CT and mathematics content, paying specific attention to teachers' articulated goal.

The teachers in our research group were able to successfully design the CT-integration activities. To illustrate this, we present a case from one curriculum designed by Ada (pseudonym), which was a general elementary mathematics activity for middle-income students in $4^{th}$ grade.

A case study (Yin, 2011) was produced to investigate the Computational Thinking skills in the research. This strategy provides a detailed method for systematically studying and informing how CT ideas and practices are enacted. These include how CT provides opportunities for students to solve problems; how students cultivate problem-solving skills in every step; how the rules of CT are applied to the practice in both mathematics teaching and learning.

## Computational Thinking in mathematics problem-solving

## Question

*Basic Question, from 2.1 – 2.4*: There are a total of 12 chickens and rabbits in one cage and a total of 36 legs. How many rabbits are there in the cage?*

<u>*The solutions and explanations for the Basic question are described in sections 2.1 to 2.4</u>

*Complex Question, from 2.5–2.7*: A courier delivers 500 glasses to a company. Both parties agree that the shipping fee is 2 cents per glass and if the courier damages one piece, not only will he not get the shipping cost, but the courier must also compensate the company 8$ per damaged glass. At the final settlement, the courier received a total of 95 dollars for shipping. How many glasses were damaged?*

<u>*The solutions and explanation for the complex question are described in sections 2.5 to 2.6</u>

## Goal of the activity

Students learn how to use <u>***hypothetico-deductive method***</u> (HDM) to solve mathematics problems. The HDM is a proposed description of the scientific method. According to this method, scientific inquiry proceeds by formulating a hypothesis in a form that can be falsifiable, using a test on observable data where the outcome is not yet known. The goal is to cultivate students' problem-solving abilities while connecting general knowledge to mathematics education. In the problem-solving process, students need to make assumptions based on prior knowledge of daily life and try to figure out the numeral relationship between different objects within mathematics problems, thus getting the results and finally applying the problem-solving patterns into complex problems.

## How to teach it? (i.e., the teachers' intention to see "how" the students' CT skills could be cultivated through the mathematics PS by designing the activity)

1. Encourage students to activate prior knowledge and connect with the information provided by the problem by posing questions like, "How many feet does a rabbit or chicken have?" "According to the question, what information are you provided with?"

   <u>***CT Generalization of prior knowledge***</u>

   > We identify the prior knowledge (data) we already know through our general knowledge—A rabbit has four legs, a chicken has two legs—and connect this data to the question.

   <u>***Abstraction of information***</u>

   > As we move from the generalization of prior knowledge, we transition towards identifying the algorithmic relationships in the problem—animal number: rabbit + chicken = 12; legs: rabbit + chicken = 36.

2. Based on this information, we can deduce possibilities of how many rabbits or chickens may be in the cage.

**Table 2: 6 rabbits and 6 chickens in one cage**

(**Table method)**

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Rabbit | 12 | 11 | 10 | 9 | 8 | 7 | **6** | 5 | 4 | 3 | 2 | 1 | 0 |
| Chicken | 0 | 1 | 2 | 3 | 4 | 5 | **6** | 7 | 8 | 9 | 10 | 11 | 12 |
| Legs | 48 | 46 | 44 | 42 | 40 | 38 | **36** | 34 | 32 | 30 | 28 | 26 | 24 |

⇒ *"If there are all rabbits, how many legs are there in the cage?"*

**Table 3: 12 rabbits and 0 chickens in one cage**

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Rabbit | **12** | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
| Chicken | **0** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| Legs | **48** | 46 | 44 | 42 | 40 | 38 | 36 | 34 | 32 | 30 | 28 | 26 | 24 |

⇒ *"If there are all chickens, how many legs are there in the cage?"*

**Table 4: 12 chickens and 0 rabbits in one cage**

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Rabbit | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | **0** |
| Chicken | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | **12** |
| Legs | 48 | 46 | 44 | 42 | 40 | 38 | 36 | 34 | 32 | 30 | 28 | 26 | **24** |

⇒ "Every time you consider a rabbit as a chicken, how many feet are there?"

⇒ "What is the relationship in the differences observed between the number of chickens or rabbits and the number of feet?"

Explanation 1: It means that every time we consider a two-leg chicken as a four-leg rabbit, we count two feet less.

Explanation 2: It means that every time we consider a four-leg rabbit as a two-leg chicken, we count two feet more.

Using these algorithmic relationships to solve practical problems is an important way to cultivate students' problem-solving abilities.

***Decomposition of a problem***

Identify every step necessary to understand and accomplish the whole task by building up algorithmic relationships, such as proposing the algorithmic relationship between Rabbit-leg and Chicken-leg.

***Debugging of the errors***

Debugging here is the process of finding and fixing errors in making the tables. For instance, in section 2.1, when drawing the tables, the students need to test from "if I have 12 rabbits and 0 chicken, how many legs will I get?" to "if I have 0 rabbits and 12 chickens, how many legs will I get?" The debugging process involves testing "what if I have 12 rabbits?" "what if I have 11 rabbits?" "what if I have 10 rabbits?"… and finally reach to the right answer.

*List the formulas*

⇒ Assume that there are only chickens in the cage

- 8 x 2 = 16 (If all the rabbits are chickens, the total is 16 legs)
- 26 – 16 = 10 (If rabbits are chickens, rabbits with 4 feet are counted as chickens with 2 feet. Each rabbit has 2 fewer feet, and 10 feet is the number of feet of the lesser rabbit.)
- 4 – 2 = 2 (If it's all chickens, that means a rabbit with 4 legs is a chicken with 2 legs. So, 4 – 2 means that if a rabbit is regarded as a chicken, 2 feet should be counted less.)
- 10 ÷ 2 = 5 (How many rabbits are considered as chickens? Are 10 feet missing? Just look at how many 2s are in 10—that is to count the rabbits as chickens, so 10 ÷ 2 = 5, which is the number of rabbits.)
- 8 – 5 = 3 (The total number of chickens and rabbits minus the number of rabbits is the number of chickens, 8 – 5 = 3 chickens.)

***Decomposition of problem procedures***

Identify every step necessary to solve the problem.

*Propose the hypothetical method for the question*

Just now we solved Case 1 by assuming that the animals were all chickens or all rabbits, and this method is called the hypothesis method. This is a basic method to solve the problem of "chicken and rabbit in the same cage." It is also a more common general method in arithmetic methods.

**Design intent:** The arithmetic of the hypothesis method is difficult for most students to understand and master. Using the table method, the combination of numbers and shapes guides students to explain the arithmetic more completely and accurately according to the diagram, learn to think strategically, and learn to explain effectively, which can make students experience the advantages of the hypothesis method more intuitively.

***Generalization of a solution***

Generalize the solution as "Pattern" to solve similar questions in the future.

*Propose the complex question: A courier delivers 500 glasses to a company. Both parties agree that the shipping fee is $2 per glass and if the courier damages one piece, not only will he not get the shipping cost, but the courier must also compensate the company 8$ per damaged glass. At the final settlement, the courier received a total of 95 dollars for shipping. How many glasses were damaged?*

This type of questions aligns with the strand of financial education in mathematics and allows the use of the "hypothetical method."

⇒ If all glasses are intact

- 500 x 2 = 1000 (If all the glasses are intact, the courier will get 1,000 dollars)
- 1000 – 950 = 50 (If broken glasses are intact glasses, 50 dollars is the money of compensation and shipping fees for broken glasses.)
- 2 + 8 = 10 (Assuming it's all intact glasses to be delivered. So, 2 + 8 = 10 means that if we regard the broken glasses as the intact ones, 10 dollars should be calculated less.)

- 50 ÷ 10 = 5 (50 ÷ 10 = 5 is the number of broken glasses)
- 500 – 5 = 495 (the number of intact glasses)

***Generalize a solution from one basic problem to another***

Recognize the pattern from a "simple" pattern into complex questions.

Connecting the daily-life problems (e.g., connecting the financial problems with mathematics) with the "chicken and rabbits in the same cage" problem introduces the hypothetical method and guides students to deepen their understanding of algorithmic relationship.

The procedures applied can serve as pilots to enhance CT. These contribute to students' visualization of the procedures regarding problem-solving, thus simplifying the mathematical relationships involved in the procedures.

Hypothesizing and generalizing use the same method to solve the problems but involve different variables. Thus, the generalization cannot be applied immediately, and our understanding of the problem cannot be simplified by simple calculations before considering all the data. Therefore, we use abstraction and decomposition to reconstruct the algorithms and patterns offered in the simple question to better understand the mathematical relationships and complete the problem-solving. As such, we can design a fresh arithmetic solution. In the process, students are encouraged to design, build, and assess solutions for themselves rather than using the existing solutions and prior algorithmic relationships. This automation inspired by the problem-solving process is key to developing CT skills, which can be commonly found and applied to other procedures of problem-solving within mathematics education.

## Extend CT to complex problem solving

While solving complex problems provided by the learning activities in class, there is often much ambiguity within the solving process. This ambiguity arises from the need to recognize patterns and relationships within the mathematics problem-solving process (Kellman et al., 2010). Additionally, the diverse relationships between different methods or different questions make it difficult for elementary students (i.e., From K-1 to K-6) to construct patterns since the elementary level is a pathway for students to progress from specific questions to abstract questions (Greeno, 2021). In this "rabbit – chicken – leg" activity designed by the teachers, complexity is reflected in the generalization of pattern usage. Compared to the simple question in the activity, the complex one requires students to reconsider their prior knowledge and algorithmic relationships for calculating.

The definition and application of CT do not come directly from the solving process and skills, but rather from the cultivation of thinking at the cognitive level (Tsarava et al., 2022). The hypothetical solutions used in mathematical problem solving are composed of generalization, abstraction, decomposition, and other techniques. However, even simple data provided through the problem can reveal diverse methods for arriving at the answers. The relationships and conditions that we need to consider are distinctive of CT in mathematics. CT provides efficiency, consistency in procedures, and accuracy while also providing students with an orderly and cultivating atmosphere to connect mathematics problems to daily life (Shute et al., 2017). Thus, for teaching problem solving within mathematics, we can find the traces of CT and explore the potency of CT.

Grover and Pea (2013) specify that extensive research over decades has focused on issues related to teaching and learning skills, concepts, and practices relevant to CT. Therefore, researchers should not only discover the potency of CT in mathematics classrooms but also investigate the traces and potency of CT in other disciplines. By doing so, we can avoid constricting the implementation of CT to one classroom or school subject and instead see the efficiency and influences of CT across various areas.

## Conclusions

In this study, we observed and analyzed mathematics problem-solving procedures at the 4th grade level to inform how CT ideas and practices are enacted. This included how CT can be leveraged to provide opportunities for students to apply their problem-solving skills efficiently and how these skills can be cultivated at different stages.

**RQ1)** Practically, findings from this study highlighted how CT can be displayed in both mathematics teaching and learning, as well as the challenges associated with the latter. It seems the ability to embrace CT is crucial for mathematics problem-solving, especially when planning to teach problem-solving.

**RQ2)** Practicing the application of CT in problem-solving learning situations in mathematics could potentially facilitate the transfer of these skills in other disciplines. The ability to embrace CT is crucial for mathematics problem-solving, as it can effectively guide the preparation of procedural instructions regarding the development of mathematical skills and understanding. The awareness and application of CT in mathematics problem-solving process could enhance students' efficiency in solving more complex and diverse problems, thus improving their problem-solving skills in various disciplines outside of mathematics (e.g., literacy, science, arts, etc.).

The integration of CT into mathematics education has, more often than not, been ignored, since CT has not been well-defined in specific mathematics teaching and learning (Selby & Woollard, 2013). Furthermore, teachers have no access to the resources of CT (Rich et al., 2020). However, CT provides an opportunity for teachers to instruct the students procedurally and creatively. Teachers could introduce digital devices into every step of the solving process (e.g., use computers to test the results by inputting the numbers of the rabbits/chicken). Besides, if other aspects of CT can be integrated into the classrooms, after considering social needs, teachers' competencies, and students' skills, CT can have more meaning for mathematics education and even other disciplines (Bundy, 2007; Rich et al., 2019).

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