What is geometric about geometric sequences? Informal justifications for mathematical terminology

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Communication in mathematics sometimes leverages terminology that has different meaning outside of the mathematical register. For example, what is irrational about an irrational number? What is perfect about a perfect number? In the former case, a definition clarifies any confusion: a number is irrational in the sense that it cannot be written as a ratio (under some constraints). In the latter case, even the definition does not help; the fact that a perfect number is the sum of its proper divisors does not appear inherently "perfect" in a colloquial sense.

In such situations, exploring the name of an object that seems inexplicable at first blush can reveal interesting insight into mathematical history and convention, including how ideas have developed and changed over time. It is this observation that led to the present study, in which we asked teachers to consider what is "geometric" about the geometric sequence. By attempting to answer this question, we hoped that these teachers (and later, the students that they teach) would discover unexpected historical connections between areas of mathematics that they had previously considered distinct. It is empowering for teachers to understand mathematics as the cumulative result of centuries of human effort—even though, as a result, it may sometimes seem disorganized or arbitrary.

The participants in this study were comprised of both prospective teachers ($n = 9$) and practicing teachers ($n = 15$). The prospective teachers were enrolled in a mathematical problem solving course in their last term of a teacher certification program. The practicing teachers were enrolled in a professional development course that focused on learning and deepening mathematical knowledge in the context of pedagogy. Both groups were given the geometric sequences task (Figure 1), a kind of scripting task.

> You are starting a unit on geometric sequences. After you have provided several examples, you are faced with a student's question.
>
> **Student: What is GEOMETRIC about geometric sequences?**
> **Teacher: What do you mean?**
> **Student: You called these sequences "geometric", but these are just sequences of numbers…**
> **Teacher: …**
>
> YOUR TASK is to develop an imaginary dialogue in which the student question is discussed and to justify your course of action.

Figure 1: The prompt for the geometric sequences task.

Scripting tasks were originally conceptualized as lesson plays, a more robust form of lesson planning that tasks educators to envision hypothetical dialogues that would serve a key role in their lessons (Zazkis et al., 2009; Zazkis, et al., 2013). More recently, scripting tasks have evolved to encompass any written dialogue in a mathematical context (e.g. Marmur & Zazkis, 2018; Kontorovich & Zazkis, 2016; Zazkis et al. 2013). In addition to the scripting portion of the task pictured in Figure 1, participants in the study were also asked to explain the actions taken by the characters in the script and how their personal understanding of the mathematics might have differed from what was presented in the script.

In analyzing the received submissions, the research team recognized the ability of Toulmin model of informal argumentation (Toulmin, 1958/2003) to organize and make sense of the structure of the scripts. A Toulmin model is traditionally comprised of *data* in support of a *conclusion*; the arguer may use a *modal qualifier* (e.g., "so, it is probably the case that…") in order to express the degree of confidence with which they believe the conclusion follows from the data. In addition, both *warrants* and *backing* may present reasons why the data supports the conclusion; *rebuttals* may present reasons why it does not.

Because the teacher-character in the submitted scripts is trying to justify a naming convention and not a mathematical principle, their arguments were often informal and thus allowed for the use of Toulmin's model. However, their arguments were uniquely structured in such a way that required us to modify the model: the teacher-characters were not arguing that a conclusion followed from some data, but that some data was probably used to draw a forgone conclusion. That is, the fact that geometric sequences are named as such cannot be disputed; the question, then, is: "What historical precedence is there for naming geometric sequences in this way?" We represent this novel form of argumentation by conceptualizing Toulmin-R models, which reverse the direction of the argument to focus on discovering a source of data. Toulmin-R models, like the usual models, still allow for rebuttals, warrants, and backings. As we discovered in the data, student-characters often voiced rebuttals aimed at the teacher-characters' choices of data. The teacher-characters then defended their choices with warrants.

The submitted scripts for the geometric sequences task revealed that participants often attributed the naming of geometric sequences to historical people or communities. Greek scholars (and the work they did with sequences and geometry) prevailed as an especially popular and compelling choice of data. Participants considered the etymology of the words *arithmetic* and *geometric*, common or challenging geometric problems of the age, and the fascination that Greek scholars had with sequences in general when composing their arguments.

In particular, Euclid's Elements (2002) arose in several different submissions, where it manifested as three distinct warrants. First, the continued proportions that Euclid considers in Book VIII closely mirror finite geometric sequences. Next, a participant noted that Euclid constructs a mean proportional in Proposition 5 of Book VI by inscribing a right triangle along the diameter of a circle. Finally, some participants reference Proposition 14 of Book II, wherein Euclid constructs a square with the same area as a given "rectilinear figure," i.e. rectangle. The backing for each of these warrants leveraged the geometric nature of line segments, triangles, and quadrilaterals respectively.

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